Preface

For more than two score years I have meditated upon the nature of Mathematics, upon its significance in Thought, and upon its bearings on human Life. In the following course of lectures I have endeavored to present, in the language current among educated men and women, some of the maturer fruits of that study.

Though the course is designed primarily for students whose major interest is in Philosophy, I venture to hope that the lectures may not be ungrateful to a much wider circle of readers and scholars:

To the growing class of such professional mathematicians as are not without interest in the philosophical aspects of their science.

To the growing class of such teachers of mathematics as endeavor to make the spirit of their subject dominate its technique.

To the growing class of those natural-science students who are interested in the logical structure and the distinctive method of mathematics regarded not only as a powerful instrument for natural science but also and especially as the prototype which every branch of science approximates in proportion as its basal assumptions and concepts become clearly defined.

To the innumerous but precious tribe of those literary critics who know that the art of Criticism owes its first allegiance to the eternal laws of thought.

To such psychologists as are interested either in the psychology of mathematics or in the mathematics of psychology.

To such sociologists as desire to conceive the nature of our humankind justly—in accord with the mathematical principle of "logical types" or dimensionality.

To the rapidly increasing class of engineers who are learning to conceive engineering worthily, as the science and art of directing the civilizing energies of the world to the advancement of the welfare of all mankind including posterity.

Finally, to all readers who desire to acquire a fair understanding of such genuinely great mathematical ideas as are accessible to all educated laymen and to come thus into touch with the universal spirit of the science which Plato called divine.

In closing this preface I desire to record my gratitude to Mr. John Macrae, vice-president of E. P. Dutton & Company, for his generous encouragement in this enterprise.

Cassius Jackson Keyser.

Columbia University,
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Lecture I

Introduction

INTELLECTUAL FREEDOM AND LOGICAL FATE—MATHEMATICAL OBLIGATIONS OF PHILOSOPHY AND EDUCATION—COMMON HUMANITY AND INDIVIDUALITY—HUMANISTIC AND INDUSTRIAL EDUCATION—MAN NOT AN ANIMAL—ETHICS NOT A BRANCH OF ZOOLOGY—EXCELLENCE AND THE MUSES—LOGIC THE MUSE OF THOUGHT—THE HEROIC TRADITION IN PHILOSOPHY—RADIANT ASPECTS OF AN OVER-WORLD.

It is the aim of the following lectures to point out, in a manner suitable for you as students of Thought, and to submit for your consideration, some of the more essential and more significant relations between Mathematics and Philosophy. Each of these great terms is to be understood in its most embracing sense. Mathematicians sometimes speak contemptuously of philosophy; and philosophers sometimes speak contemptuously of mathematics. The contempt thus manifested does not spring from mathematics in the former case, nor from philosophy in the latter; in both cases it springs out of ignorance—philosophical ignorance of mathematicians and mathematical ignorance of philosophers. No doubt philosophically unenlightened mathematicians and mathematically unenlightened philosophers will quarrel in the future as in the past; but in the future as in the past, the quarreling and the sneering will be the quarreling and sneering of men and not of the great subjects they represent and misrepresent; for between the spirit of mathematics and the spirit of philosophy there is no discord, no antagonism, no strife; they are by their natures friendly rivals in the pursuit of truth and light; they are companions in excellence; they are comrades in the service of wisdom.

I have said that the "aim" of these lectures is to disclose fundamental connections between mathematics and philosophy. What I have described as their "aim" is not so much the aim, or end, as a means. For it will become increasingly evident as we advance that the work we are to be engaged in is fundamentally the study of Fate and Freedom—logical fate and intellectual freedom. I mention the matter here because you ought to have it consciously in mind from the beginning. You should bear it in mind at every stage of the discussion, even in connections where so warm an interest may seem remote. A preliminary word of explanation is therefore desirable.

We are going to deal with ideas—with their characters, with their meanings, with their relations. Now, an idea is in itself an eternal thing and the relations of an idea with other ideas are eternal. An idea is just what it is and it is unalterable; a relation among ideas is just what it is and it is unalterable. We do, indeed, often speak as if such were not the case; we habitually speak as if ideas and their relations were temporal affairs, impermanent, mutable, malleable, capable of growth, of modifcation, of decay, of destruction, as when we say, for example, that we have "changed" our ideas or that such-and-such an idea has "grown" in importance or has "become" sterile or is "dead." It is, I fancy, hardly necessary to say that all such ways of speaking are figurative,—convenient no doubt, often pleasing, sometimes very effective, yet thoroughly figurative,—and that, if taken literally, they quickly and inevitably lead to scientific and philosophic disaster. You or I may abandon an idea that we have held and we may adopt an idea that is new to us; the "old" one and the "new" one may closely resemble each other; they may indeed be identical in some respect and may even be called by the same name; but neither of them has been transmuted into the other; each of them remains and will remain just what it was. Let me illustrate the eternality of ideas and of their relations by means of a simple example. You know that in discourse ideas are represented by symbols—by words or other signs. Consider the symbols 2, 7, 9, +, and =; each of them stands for an idea familiar to all of us. The symbols are man-made; but the things they stand for, though they were discovered by man, are not man-made; they are increate, as Milton would say, and indestructible, and the like is true of their relations; one of these is expressed by the statement (a) $2+7=9$; the statement expressing the relation is a creature of man, but the relation itself is not—man discovered it, but he did not make it—it is a thing increate and indestructible, the same yesterday, today and forever. The truth of what I have just now said is very evident, but the illustration is arithmetical. Is the eternality equally evident in the case of all other ideas and their relations? No, it is not equally evident, but it is none the less true. Shall we take another example? Let us take one that is very far from being specifically arithmetical. Consider the statement:

(b) If something S has the property p and whatever has the property p has the property p', then S has the property p'.

You observe that the statement expresses a certain *relation* among certain ideas—the idea, for example, denoted by "something," that denoted by "property" (or quality or mark), that denoted by "whatever," and so on. The denoting terms are indeed man-made, but the ideas denoted are not, they are merely man-discovered and man-known; and the statement expressing the relation is a creature of man, but the relation itself, though man discovered it, was not created by him: it is an unoriginated thing, immutable, universal, timeless. The illustration is very general, very abstract and very cold. Perhaps you prefer something warmer, more specific, more concrete. Well, it is easy to find such, for the foregoing general statement is infinitely rich in concrete applications. Let me instance one of them, one that is sufficiently warm; it is indeed one that goes to the very heart of our human ethics—not to our ethics as it is, but as it ought to be and as no doubt it will be. The application is this, namely:

> If human beings are by nature civilization-builders, or "time-binders," and if all time-binders, or civilization-builders, are both inheritors from the toil of by-gone generations and trustees for the generations to come, then we humans stand in the double relationship—debtors of the dead, trustees of the unborn—thus uniting past, present and future in one living, growing reality.

The infinite and eternal significance of that fact may, I trust, be left for your meditation.

Without more talk and without danger of misunderstanding, we may, I believe, now speak of ideas as constituting a world—the world of ideas. With that world all human beings as human have to deal—there is no escape; it is there and only there that foundations are found—foundations for science, foundations for philosophy, foundations for art, foundations for religion, for ethics, for government and education; it is in the world of ideas and only there that human beings as human may find principles or bases for rational theories and rational conduct of life, whether individual life or community life; choices differ but some choice of principles we must make if we are to be really human—if, that is, we are to be rational—and when we have made it, we are at once bound by a destiny of consequences beyond the power of passion or will to control or modify; another choice of principles is but the election of another destiny. The world of ideas is, you see, the empire of Fate.

Is the human Intellect, then, a slave? No: it is free; but its freedom is not absolute; it is limited by fact and by law—by the laws of thought, by the immutable characters of ideas and by their unchanging eternal relationships. Intellectual freedom is freedom to think in accord with the laws of thought, in accord with the natures of ideas, in accord with their interrelations, which are unalterable. And no variety of human freedom—no institution erected in its sacred name—if it does not conform to the eternal conditions of intellectual freedom—can stand.

What I have now said is, I hope, a sufficient preliminary intimation of what I mean by saying that our work in these lectures is to be fundamentally a study of Freedom and Fate.

Your major interest is in philosophy; mine is in mathematics. You have besides, I trust, a lively, if only a minor, interest in mathematics, as I have had from the days of my youth a genuine interest, albeit a subordinate one, in the concerns of philosophy and especially in the philosophy of mathematics. It is, I believe, a happy circumstance that your interest and mine in these great subjects are thus complementary instead of coincident or antagonistic; for in this relation of interest there is implied a corresponding relation of attainment, limitation, outlook and temper; and this relation, if we bear it in mind, will be favorable in important ways to the prosperity of our enterprise; for example, it should, on the one hand, have the effect of restraining me from adducing too lightly or too freely, with too little explanation, mathematical considerations with which you may justly feel I have no right to suppose you familiar; and, on the other hand, when you discover, as you will doubtless frequently discover, that I have fallen into error because of my philosophical limitations, it will, I hope, make you feel it your duty to "impose upon me the just retribution," in accordance with the saying of Plato that "The just retribution of him who errs is that he be set right."

It need hardly be said that no one should follow this course in the hope of thereby acquiring mathematical knowledge or skill in the usual sense of these terms. I assume that what is mainly responsible for your presence here is a desire and a hope of a different kind: you desire to gain insight into the essential nature of mathematics regarded as a distinctive type of thought; you desire to acquire knowledge of what is characteristic and fundamental in mathematical method; you hope to gain acquaintance with some of the great mathematical concepts, with such of the dominant concepts as are accessible to laymen; you desire to win a just sense of the spiritual significance of the science; in a word, your quest is for such an understanding of it as will help you to view mathematics in a vast perspective—in relation, that is, with the other sciences and arts and the other modes and forms of human activity. Such, I take it, are the ends that define our task. I should indeed be unhappy if I did not hope that the lectures, though they have been fashioned with controlling reference to the task indicated, will at the same time serve in some measure to extend your acquaintance with the existing body of mathematical doctrine. But it is to be understood that this result, if the lectures produce it, will be incidental and subsidiary to their main purpose; for they are not designed to teach a recognized *branch* of mathematics whether elementary or more advanced. Mathematical students having little or no interest in the philosophy of their science must be frankly counseled to repair to other courses for the kind of instruction they desire. And students of philosophy should not indulge themselves in the vain hope of acquiring mathematical knowledge by merely “philosophizing” about the subject or by pensively gazing upon its general aspects from an external point of view. From time immemorial, there has been but one way to become a mathematician and there will never be another: it is a way interior to the subject and involves years of assiduous toil. Short cuts to mathematical scholarship there is none, whether the seeker be a philosopher or a king.

How much mathematical training is essential to the qualification of one who may hope to follow the lectures profitably? It is natural that you should wish to ask that question at this point. The question is important and the answer easy and short: so much mathematical training—so much knowledge of algebra, geometry and trigonometry—as a capable student can acquire in one collegiate year. Compared with the existing science of mathematics such knowledge is very meagre, a bare beginning; but, taken absolutely, it is much; in respect of content or mere information as distinguished from insight and power, it is far more than Thales had, or Pythagoras or even Plato or even Galileo. It would be very convenient if I might assume more; projective geometry, for example, and some acquaintance with analytical geometry—which should remind you of Descartes, and with the calculus—which should remind you of Leibniz; for I shall be obliged occasionally to employ ideas drawn from these and other branches of mathematics, and shall have to interrupt and delay the discussion a little in order to explain the ideas as the necessity arises for using them. Perhaps I should add that, for understanding the lectures, a certain intellectual maturity, logical acumen, open-mindedness and philosophical insight are not less essential than the stated minimum of mathematical knowledge.

I desire to invite you now to a somewhat comprehensive consideration of a much larger question, one of greater difficulty and far greater importance—a question of both general and permanent interest. The question is: How much mathematical training—how much mathematical knowledge, discipline, and habit—may be reasonably regarded as indispensable to the proper equipment of a philosopher? It may well be that you will be qualified to give a better answer at the end of the course than that which I am about to submit here at the beginning of it. Nevertheless, I am disposed to think that a preliminary discussion of the matter will be of some service. A complete discussion would involve many considerations differing greatly in weight. I shall ask your attention to such of them as seem to me cardinal and decisive.

The first consideration grows out of the fact that a philosopher is a human being. It is immediately evident that the proper equipment of a philosopher must include as much mathematical training as is essential to the appropriate education of men and women as human beings. How much is that? Be good enough to note what the question precisely is. I am not asking how much mathematical discipline is essential to a "liberal education" for this fine term, though clearly defined long ago by Aristotle in terms of spiritual interest and attitude, has in our day lost its significance even for the majority of academic folk, who ought to be ashamed of the fact. That great man, the late Lord Kelvin, used to tell his students that among the "essentials of a liberal education is mastery of Newton's *Principia* and Herschel's *Astronomy*." On the other hand, such educators as Matthew Arnold, John Henry Newman, Thomas Huxley, though differing infinitely in their outlooks upon the world and in their estimates of worth, yet unite in denying Kelvin's contention impetuously or even with scorn. Let us so frame our question as to avoid that debate. The question is: How much mathematical discipline is essential to the appropriate education of men and women as human beings? This exceedingly important question admits of a definite answer and it admits of it in terms of a supremely important and incontestable general principle. A clue to the principle is found in the phrase I have just now employed: education of men and women *as human beings.* Before stating the principle, it will be convenient to give it a name. I shall call it the Principle of Humanistic Education as distinguished from what has come to be designated in our day as Industrial Education. I say "as distinguished from" because the two varieties of education, whether they be compared with respect to the conceptions which lie at the heart of them or with respect to the motives which actuate and sustain them, are widely different. In order to set the principle in a clear light, let me indicate briefly the obvious facts lying at its base and leading naturally to its formulation.

What the individuals composing our race have in common falls into two parts: a part consisting of those numerous instincts, impulses, traits, propensities and powers which we humans have in common, not only with one another, but with many of the creatures constituting the world of animals—a subhuman [1] world; and a second part consisting of such instincts, impulses, traits, propensities and powers as are distinctively human. These latter, we may say, constitute our Common Humanity. They present, indeed, an endless variety of detail, but in the long course of man's experience with man he has learned to group them, in accordance with their principal aspects, into a small number of familiar classes. And accordingly, the nature of our common humanity is fairly well characterized by saying that human beings as such possess in some recognizable measure such marks as the following: a sense for language, for expression in speech—the literary faculty; a sense for the past, for the value of experience—the historical faculty; a sense for the future, for prediction, for natural law—the scientific faculty; a sense for fellowship, coöperation, and justice—the political faculty; a sense for the beautiful—the artistic faculty; a sense for logic, for rigorous thinking—the mathematical faculty; a sense for wisdom, for world harmony, for cosmic understanding—

1. See Lecture XX for a discussion of Korzybski's concept of Man in terms of Time—a conception according to which humans are not animals.

the philosophical faculty; and a sense for the mystery of divinity—the religious faculty.

Such are the evident tokens and the cardinal constituents of that which in human beings is human. It is essential to note that to each of the senses or faculties in virtue of which humans are, not animals, but a higher class of beings, there corresponds a certain type of distinctively human activity—a kind of activity in which all human beings, whatever their stations or occupations, are as humans obliged to participate. Like the faculties to which they correspond, these types of activity, though they are interrelated, are yet distinct. Each of them has a character of its own. Above each of the types there hovers a guardian angel—an ideal of excellence—wooing our loyalty with a benignant influence superior to every compulsive force and every authority that may command. Nothing more precious can enter a human life than a vision of these angels, and it is the revealing of them that humanistic education has for its function and its aim. Stated in abstract terms the principle is this. Each of the great types of distinctively human activity owns an appropriate standard of excellence; it is the aim of humanistic education to lead the student into a clear knowledge of these standards and to give him a vivid and abiding sense of their *authority* in the conduct of life. Ethics is not a branch of Zoology.

It is plain that this conception stands in sharp contrast with the central idea of industrial education. For humanistic education has for its aim, as I have said, the attainment of excellence in the things which constitute our *common* humanity. On the other hand, industrial education is directly and primarily concerned with our *individualities.* It might, therefore, be more appropriately called individualistic education. It regards the world as an immense camp of industries where endlessly diversified occupations call for special propensities, gifts, and training. Accordingly its aim, its ideal, is to detect in each youth as early as may be the presence of such gifts and propensities as tend to indicate and to qualify him for some specific form of calling or bread-winning craft; then to counsel and guide him in the direction thereof; and finally, by way of education, to teach him those things which, in the honorable sense of the phrase, constitute the tricks of the trade.

What are we to say of it? The answer is obvious. Industrial education, rightly conceived, is essentially compatible with the humanistic type; it may breathe the humanistic spirit; the two varieties of education are essential to constitute an ideal whole, for human beings possess both individuality and the common humanity of man. Industrial education, when thus regarded as supplementary to humanistic education, is highly commendable; but when it is viewed as an equivalent for the latter or as an ideal substitute for it, it is ridiculous, contemptible and vicious. For the fact must not be concealed that a species of education which, in producing the craftsman, neglects the man, is, in point of kind and principle, precisely on a level with that sort of training which teaches the monkey and the bear to ride a bicycle, or the seal to balance a staff upon its nose or to twirl a disc.

These considerations are no doubt obvious. I should not dwell upon them at so great length but for the fact that in the excitement and confusion of our industrial age the most obvious of important facts and the most evident of important principles are so commonly lost sight of that they require to be cited again and again and again. Nowhere is the confusion of the time more evident than in the somewhat noisy and sometimes acrimonious discussion that has been recently and still is going on throughout our country regarding the value of mathematics as a subject in secondary and collegiate education. The instigators of the discussion, those, that is, who advocate so reducing mathematical requirements as practically to abolish the subject from curricula of general education, are not malicious nor insincere; many of them, I do not doubt, are well-meaning citizens. And if their rather voluminous discourses are often singularly lacking in coherence, in clarity and in depth, the defects are not due to evil intentions but rather, I suspect, to confusion and a lack of just that sort of discipline which the subject the authors are engaged in depreciating is peculiarly qualified to give. Perhaps we should not be astonished. If the saying of Sir Oliver Lodge be true that “the mathematical ignorance of the average educated person has always been complete and shameless,” one ought not, I suppose, to be too much astonished if in a vast, crude, formless, sprawling democracy like ours, a way to educational “leadership” is sometimes found by men whose innocence, not only of mathematics but of the other great subjects, including the principles of education, is well-nigh complete and shameless. And yet, despite familiarity with the phenomenon, it is sometimes a bit hard to avoid astonishment and even a loss of patience. Not long ago a high-placed counselor of a well-known college of liberal arts challenged me, with defiant confidence and unfeigned solemnity, to give any good reason why college students should be required to pursue a course in algebra rather than one in some practical art, say the art of cooking mutton chops. On receiving such a challenge from a grown man, what should a grown man do? Confess his astonishment? Betray an exhaustion of patience? Fly to the easy refuge of ridicule? Any such reaction would probably have been misunderstood. In dealing with a solemn question, no matter how stupid, it is usually the wiser course to treat it with respect if possible. I might have responded, in the fine words of Professor Whitehead,[2] that

> "Algebra is the intellectual instrument which has been created for rendering clear the quantitative aspects of the world. … Through and through the world is infected with quantity. To talk sense, is to talk in quantities. It is no use saying that the nation is large,—How large? It is no use saying that radium is scarce,—How scarce? You can not evade quantity. You may fly to poetry and to music, and quantity and number will face you in your rhythms and your octaves. Elegant intellects which despise the theory of quantity are but half developed. They are more to be pitied than blamed."

It did not seem to me, however, that one capable of issuing such a challenge as that to which I have alluded could feel the weight of such a response, and I did not make it. It is, you observe, a response in terms of *quantity.* Quantity is indeed omnipresent in our world; but so, too, is *quality,* and of the two things, the latter is perhaps the more universal in its appeal. Algebra is indeed essential to the theory of quantity and the theory of quantity is essential to the subjugation of natural resources to the use of man; of quality, on the other hand, algebra is not a science but, though it is not a science of quality, it *has* a quality, a human quality, to which it owes its high rank in the spiritual hierarchy of human disciplines. And so I endeavored, with poor success I fear, to answer the challenge in terms of quality. I invoked the principle which in this lecture I have been calling the principle of humanistic education. I sought, that is, to make it clear that, in contrast with the practical arts, the science of algebra as a discipline possesses a certain quality by virtue of which, if the subject be rightly administered, the student is gradually brought into the presence of one of those great standards of excellence by which, as we have seen, distinctively human activity in all its principal types is

2. A. N. Whitehead: *The Organization of Thought.* Cambridge University Press.

to be guided and judged. The standard to which I refer is, as you have doubtless surmised, the standard of excellence in the quality of thinking as thinking—the standard which mathematicians are accustomed to call Logical Rigor—clarity, that is, precision and coherence.

And now the mention of that great term may serve to reassure you, should you have begun to suspect that in the course of this rather long excursion I may have forgotten the question initiating it. The question is. How much mathematical training is essential to the appropriate education of men and women as human beings? I have said that the question admits of a definite answer in terms of a supreme and incontestable principle. I have stated the principle as well as I can and have tried to signalize its importance for a general theory of education. It remains to apply it to the specific question before us. The task is not difficult. It is plain that one of the great types of distinctively human activity—perhaps the greatest and most distinctively human type—is what is known as Thinking—the handling of ideas as ideas—the formation of concepts, the combination of concepts into higher and higher ones, discernment of the relations subsisting among them, embodiment of these relations in the forms of judgments or propositions, the ordering and use of these in the construction of doctrine regarding life and the world—in a word, the whole complex of activity involved in the discourse of Thought. It is essential to the argument I am making to keep steadily in mind that this kind of activity, our sense for it, our faculty for it, the need to which it ministers, the joy it gives, and the obligation it imposes are part and parcel of what we have been calling our common humanity as distinguished, on the one hand, from that which is animal in man, and, on the other, from such special propensities or other marks as give the differing specimens of human-kind their respective individualities. Thinking is not indeed essential to life, but it is essential to human life. All men and women as human beings are inhabitants of the *Gedankenwelt*—citizens, so to speak, of the world of ideas, native citizens of the world of thought. And now what shall we say is the prototype of excellence in thinking? What is the hovering angel wooing our loyalty to what is best in thinking? What is the muse of life in the world of ideas? An austere goddess, high, pure, serene, cold towards human frailty, demanding perfect precision of ideas, perfect clarity of expression, and perfect allegiance to the eternal laws of thought. In mathematics the name of the muse is familiar: it is Rigor—Logical Rigor, which signifies a kind of silent music, the still harmony of ideas, the intellect's dream of logical perfection.

Can the dream be realized? I am well aware that most of the things which constitute the subject-matter of our human thinking—that most of the things to which our thought is drawn by interest or driven by the exigencies of life—are naturally so nebulous, so vague, so indeterminate that they cannot be handled in strict accordance with the rigorous demands of logic. I am aware that these demands can not be *fully* satisfied even in mathematics, the logical science par excellence. Nevertheless I contend that, as the *ideal* of excellence in thinking, Logical Rigor is supremely important, not only in mathematical thinking, but in all thinking and especially in just those subjects where precision is least attainable. For without this ideal, thinking is without a just standard for self-criticism, and without light upon its course; it is a wanderer, like a vessel at sea without compass or star. Were it necessary, how easy it would unfortunately be to cite endless examples of such thinking from the multitudinous writings of our time. Indeed, if the pretentious books produced in these troubled years by men without logical insight or a sense of logical obligation were gathered into a heap and burned, they would thus produce, in the form of a bright bonfire the only light they are qualified to give. "Logic," it has been said, "is the child of a good heart and a clear head." We know, however, that an evil heart is not essential to a fool and that, on the other hand, few heads are naturally so clear as not to require discipline.

Now, it so happens that the term mathematics is the name of that discipline which, because it attains more nearly than any other to the level of logical rigor, is better qualified than any other to reveal the prototype of what is best in the quality of thinking as thinking. And so, in accordance with the principle of humanistic education, we have to say that the amount of mathematical training essential to the appropriate education of men and women as human beings and essential, therefore, to philosophers as human beings, is the amount necessary to give them a fair understanding of Rigor as the standard of logical rectitude and therewith, if it may be, the spirit of loyalty to the ideal of excellence in the quality of thought as thought.

Such is my answer to the question that has detained us so long. It is, you observe, a qualitative answer in terms of a great ideal and a sovereign principle of education. If I must add a word touching the strictly quantitative aspect of the question, if I must, that is, attempt to indicate the extent of courses and the length of time necessary and sufficient to yield the required quality and degree of training, I do so with less confidence and far less interest. For so much, so very much, depends on the pupil's talent and the quality of instruction. A considerable degree of native mathematical ability is much more common than is commonly supposed. Born mathematical imbeciles are rare. Youth of fair mathematical talent constitute an immense majority. I venture to say, regarding the question of time and the extent of courses, that, for pupils of fair mathematical endowment, a collegiate freshman year or even a high school senior year of geometry and algebra, if the subjects be administered in the true mathematical spirit, with due regard to precision of ideas and to the exquisite beauty of perfect demonstration, is sufficient to give a fair vision of the ideal and standard of sound thinking.

Herewith, I have come to the end of what I desired to say respecting the mathematical equipment essential to a philosopher in so far as its measure depends upon the fact that philosophers are human beings. It remains to enquire what further mathematical attainments are to be regarded indispensable to the proper equipment of a philosopher as a philosopher. It is evident that the answer must be sought in the nature of the philosopher's vocation. It would be presumptuous in me, a student of mathematics, to offer to teach you, who are students of philosophy, the nature of your vocation, but I may remind you of it for it is necessary to have it clearly in mind if we are to see its bearings upon the question in hand. No one, I suppose, has conceived the philosopher's vocation more justly and nobly or characterized it more clearly and truly than Plato, as no other has drawn, with such clarity and charm, with so perfect a union of finesse and amplitude, so beautifully and so truly, the spiritual portrait of the genuine philosopher. You are, of course, familiar with the characterization and the portrait, which together give for all time a vision of the great ideal: what genuine philosophy is, and the philosopher ought to be. I wish to remind you of such elements of it as our present task requires.

The genuine philosopher, says Plato, "has magnificence of mind"; there is in him "no secret corner of illiberality"; he is "noble, gracious, the friend of truth, justice, courage, temperance"; he aims at being "a spectator of all time and all existence," and so he is a lover and seeker of "wisdom," which does not consist of sense-impressions nor of "the tempers and tastes of the motley multitude" nor of fickle "blinking opinion" begotten of time-born appearances and events destined to the doom of things that perish in "the sea of change," but consists in knowledge of things that abide—of true being—of whatsoever in the world is eternal: pursuit of such wisdom is the philosopher's vocation, sustained by the twofold hope of coming at length into the full-shining presence of the Beautiful, the True, and the Good and of bringing light from them into the lives of the children of men.

From that conception of the genuine philosopher's vocation and character, what conclusion follows regarding his obligation to mathematics? An important conclusion, as I hope to show if you agree with me in thinking that we ought to ascertain what it is.

Let me say at the outset that there are two pretty obvious considerations which I do not intend to insist upon, although they are not without relevance and weight. One of them is that which conceives mathematics as being itself a branch of philosophy; the other relates to the familiar contention of Plato, that mathematical discipline is indispensable as a preparation for what he conceived to be the philosopher's distinctive task—that of Dialectic.

As to the former consideration, one might argue, pertinently and confidently, that both historically and in accordance with the foregoing conception of philosophy, Logic is one of its branches; that mathematics not only employs logic as an instrument but is, in fact, identical with it, mathematics (as traditionally viewed) being related to logic (as traditionally viewed) as the trunk and branches of a tree are related to its roots; that, consequently, mathematics, being identical with logic, is not external to philosophy but is, strictly speaking, one of its principal divisions; and that, accordingly, philosophers, if they are not to be ignorant of one of the chief departments of their own subject, are obliged to be, not merely mathematical *dilettanti,* but mathematical students, serious explorers of the science. Theoretically, the argument is sound, which is the highest quality of argument as such. I do not, however, as I have said, intend to press it, because it imposes on the student of philosophy an obligation that he cannot fully meet; his obligations are many, too many and too great; he may not reasonably hope to win the proper competence of a mathematician in a subject where the developments, still rapidly progressing in numerous directions, have already reached proportions so great that no man, though he have the wide-reaching arms of a Henri Poincaré, can contrive to embrace them all.

Turning now to the second one of the two considerations mentioned a moment ago, let me guard against the danger of being misunderstood. You are aware that, in the view of Plato, what is peculiar to philosophy is dialectic—"the coping stone of the sciences"; you are aware that dialectic is the sole means by which the philosopher may gain a knowledge of "what each thing" in the hierarchy of being "essentially is," and by which he may gain, at length, as he ascends the scale, a vision of things supreme—absolute justice, absolute beauty, absolute truth, absolute good; you are aware that the successful employment of dialectic requires not only native "magnificence of mind," but also a long course of preparation in the subsidiary sciences; you are aware that, according to Plato, the most indispensable of these sciences are arithmetic and geometry. the former because arithmetic, not as the mere practical art of calculation but as a discipline in the logical nature of pure number, "lays hold of true being"; and the latter because "the knowledge at which geometry aims is knowledge of the eternal." Such is in brief, as you know, the famous contention of Plato respecting the importance of mathematical discipline as a preparation for philosophy. There can be no doubt that the contention is perfectly just. Why, then, do I not stress it in this connection? The reason is that the mathematical discipline insisted on by Plato is more than covered by the mathematical training I have already urged as essential to the appropriate education of the philosopher as a human being, and that we are here considering such further mathematical attainments as are essential to him as a philosopher. Before leaving this theme, however, I desire to point out a different aspect of it and in connection therewith to speak very briefly, in passing, of a matter which I have discussed elsewhere,[3] to which I hope to return at a later stage of these lectures and which, I believe, has a very important bearing upon the question before us.

After some years of reflection, I am convinced that the great Platonic Absolutes, whose "perception by pure intelligence" brings us, says Plato, to "the *end* of the intellectual world"—have indeed their proper locus *beyond* it. I am convinced that, instead of being genuine concepts amenable as such to the logical processes valid in the intellect's world, the Platonic Absolutes are radiant *ideals* of concepts, shining from above them like

3. *Science and Religion,* also *The New Infinite and the Old Theology.* Yale University Press.

downward-looking aspects of an over-world; transcending every type of excellence in which intellectual progress is possible, they appear as ideals supernal—as stars beyond the sky. I need not say that the Absolutes, thus regarded, retain their glory unimpaired and their previous value as sources of light and inspiration. We should not, however, fail to see clearly that, if they be thus regarded, the philosopher is thereby confronted by a new challenge, a new problem, a new field of study or, perhaps I should say, by an old one seen as new. For, if the Absolutes are not *in* the intellectual world but are beyond it; if they be, in fact, not concepts, but ideals of concepts, shining downward from above them, then obviously their origin, the manner and genesis of their appearance, and their significance for life, must be sought in the nature and function of that strange and familiar spiritual process omnipresent among the activities of the intellectual world and known as Idealization. And now the point I am aiming at and to which I invite your special attention is this. In the study of this great subject—the nature and function of Idealization—the philosopher and especially the theologian as philosopher—for rational theology, rightly conceived, is the science of Idealization—will have need of mathematical discipline surpassing the Platonic requirement and surpassing what I have deemed essential to the education of the philosopher as a human being. For the term "idealization" is the generic literary term for what in science and especially in mathematics is known as generalization by means of the method or process of limits. In mathematics, particularly in the modern theory of the Real Variable, in connection with the generalization of the number concept, the essential nature of Idealization, the pattern of it as the process and method of directing the attention from within a given domain of operation to the existence and the character of outlying domains, comes into perfect light. It is in mathematics and not elsewhere that Idealization is beheld in its purity; and unless the philosopher becomes familiar with it there in its purity, his endeavor to study the great process elsewhere, amid the many disguises half concealing its subtle ramifications throughout the shadowy world of general thought, will encounter serious difficulties, if not defeat.

The considerations I have now advanced, though they are subordinate, are weighty, and I commend them as worthy of your further reflection. Let us proceed, without further delay, to the heart of the matter.

We have seen that the genuine philosopher "has magnificence of mind"; that there is in him "no secret corner of illiberality"; that his vocation requires him to be "a spectator of all time and all existence"; and that the wisdom he seeks is the wisdom which consists in knowledge of whatsoever is eternal. It is these great things—the highest distinctive marks of the genuine philosopher—that determine the character of his mathematical obligations and enable us to measure them. For what is mathematics? What is that science which Plato[4] called "divine," which Goethe called "an organ of the inner higher sense," which Novalis called "the life of the gods," and which Sylvester called "the Music of Reason"? The question is not intended to call for a complete description of the science, much less for a definition of it. What it seeks is a partial description. I wish merely to draw your attention to one feature of mathematics—to that feature of it which all competent judges agree in signalizing as the chief aspect of the science viewed as an enterprise. The aspect in question I endeavored to point out some years ago in the following words. "As an enterprise, mathematics is characterized by its aim, and its aim is to think rigorously whatever is rigorously thinkable or whatever may become rigorously thinkable in course of the upward striving and refining evolution of ideas."[5] The same feafure has been recently indicated, even more clearly perhaps and somewhat poignantly, in a striking utterance by Mr. Bertrand Russell. "Pure logic, and pure mathematics (which is the same thing), aims at being true, in Leibnizian phraseology, in all possible worlds and not merely in this higgledy-piggledy job-lot of a world in which chance has imprisoned us."[6]

You know, at least in a general way, that in pursuit of that enterprise and aim through the centuries, the mathematical spirit has achieved immense results and that today the science of mathematics, as a body of permanent knowledge regarding things eternal, is a veritable continent of expanding doctrine. And so it is pertinent to ask. How can one aspiring to be a philosopher, unless he explores that growing continent of knowledge respecting what is "true of all possible worlds," be in any proper sense "a spectator of all time and all existence"? You may wish to reply that, owing to his other obligations, the philosopher cannot make the exploration fully; that indeed, owing to the nature of the continent, he cannot, without exploring it step by step, gain even so much as a clear

4. See *Memorabilia Mathematica* by Professor Moritz.
5. *Human Worth of Rigorous Thinking,* p. 3. Columbia University Press.
6. *Introduction to Mathematical Philosophy.* The Macmillan Company, New York.

knowledge of its contour and relief; that, however, notwithstanding the endless diversity of the things that are there, they have a certain essential character in common; that for the philosopher's vocation, knowledge of that common character is sufficient; and that such knowledge does not demand exploration of the continent in all its length and breadth and height and depth, but may be gained by examination of representative parts and especially of the elements which fundamentally compose the whole.

That reply, if we rightly interpret the meaning of the terms, is just. But their meaning is momentous. The mathematical knowledge which they tell us is sufficient for the purposes of the philosopher is neither slight nor simple nor easy to gain. The questions it must answer determine its nature and its scope. What are the idiosyncrasies of mathematics as a body of content? As a system of methods? As a type of activity? As a distinctive enterprise among the great kindred enterprises of the human spirit? If the science be logical, what are its relations to Logic? If it be beautiful, what are its relations to Art? If it employ hypothesis, observation and experiment, what are its relations to Natural Science? If it be purely abstract and conceptual, what are its relations to the concrete world of Sense? If it be theoretic, what are its relations to Practical Life? If it be self-critical, what are its relations to the science and art of Criticism? If it be a wisdom respecting infinite and eternal things, what are its relations to Philosophy and to Religion? If it have limitations, what are its relations to the dream of Universal Knowledge? To the challenge of these great questions and their kind, no one having "magnificence of mind," no one called to be "a spectator of all time and all existence," can fail to respond. And so we see that the mathematical obligations of the philosopher confront him with two difficult close-related Problems: the problem of *definition* and the problem of *evaluation:* he must endeavor to ascertain what mathematics essentially is and endeavor to estimate, in the terms of spiritual Worth, the rank and the dignity of the science in the hierarchy of knowledges and arts.

It is a radical error to regard these kindred tasks of definition and evaluation as belonging to the proper function of mathematicians as such. The term "mathematics" is the name of an immense class of logically related terms and most of these the mathematician must indeed define, but the term "mathematics," which names the class, is not among them; the class is not a member of itself, for no class can be; the name "mathematics" is not a mathematical term; the mathematician would be none the less a mathematician, had he never heard of it; it is a philosophical term, used by mathematicians as a convenience but never as a necessity. The proper activity, the distinctive function, of the mathematician is to mathematicize, as that of a swimmer is to swim; or that of a farmer, to farm; or that of a poet, to make poetry; or that of a trader, to trade. And it is as little the business of the mathematician to define and evaluate the peculiar type of his proper activity as it is that of the swimmer or the farmer or the poet or the trader to do the like for his. The philosopher, therefore, may not rightly look to mathematicians as such for a definition of mathematics nor for any appraisement of its significance or its worth.

Is it not true, nevertheless,—you may wish to ask—that nearly all real advancement made in the course of the centuries in these tasks of definition and appraisement has been made by mathematicians? The answer is yes, even if we do not forget or underrate the relevant contributions of Plato and Aristotle, for knowing, as they did, what was known then of mathematics, they must be counted among the mathematical scholars of their day. It must be noted, however, that, though the advancement in question was made by mathematicians, it was made by them, not in their character as mathematicians, but in their capacity as philosophers. There is nothing in the fact to astonish. For a man is greater than any occupation, and a mathematician, like a physician or lawyer or poet or statesman or farmer, may be—indeed he must be, in some measure—a philosopher as well. It is not, then, strange or a matter for wonder that there have been mathematicians who, in relation to their proper subject taken as a distinctive whole, have sometimes taken the attitude and played the rôle of philosopher. Nay, even *within* the subject, in relation to its parts, the rôle is very common; for whenever a mathematician, having acquired competence in two or more branches—say algebra and geometry—pauses to compare them, seeking to ascertain the essential nature of each, what they have in common, their respective worths and their joint significance as forms of activity, his interest and his attitude have then become for the time, whether long or short, those of the philosopher. The fact is that such minor alternations of the scientific and the philosophic interests may be constantly witnessed even in the activity of such mathematicians as ignorantly affect to spurn philosophy and to scorn its achievements; but they are not aware of it.

Of the two tasks with which, as we have seen, the mathematical obligations of the philosopher confront him, the task of definition is far more advanced than that of evaluation; and, though the work of the former is not yet complete, we know much better today what mathematics is than what it is worth. That it should be so is natural, for a just appraisement of worth depends, of course, upon the nature of the thing appraised. We are, therefore, not surprised to find that researches concerning the essential nature of mathematics have been prosecuted, especially in recent times, far more resolutely and systematically than such as aim at a critical estimate of its significance and value. In Plato and in Aristotle, as you know, research of both kinds produced results of great importance. I shall not speak of the great Greek mathematicians for their interest centered, not in the philosophy of their subject, but in the science of it. They were swimmers mainly—not non-aquatic students of swimming. It seems incredible that, after Plato and Aristotle, no important contribution to the philosophy of mathematics was made in the course of twenty hundred years. Yet that is the fact. Even the brilliant and exquisite *De L'Esprit Géometrique* of Pascal is thoroughly Aristotelian. The great revival had to await the appearance of Leibniz—of him who said, *"Ma métaphysique est toute mathématique."* As students of philosophy, you know that throughout his life this marvelous man was haunted by a magnificent dream—the dream of "a universal mathematics." In his manifold endeavors to make the dream come true is found the origin of that great critico-constructive movement which has done more than all previous centuries to disclose the essential nature of rigorous thought and which, after notable vicissitudes of fortune, is known today, in all scientific countries of the world, under the characteristic name of Symbolic Logic.

The leading names of its pioneers and contributors—Leibniz, Lambert, De Morgan, Boole, Jevons, Schröder, Peirce (C. S.), MacCall, Frege, Peano, Russell, Whitehead, Hilbert, Huntington, Couturat, and others—sufficiently indicate its international interest and the variety of genius to which it appeals. The growing literature of the subject is large. Fortunately, it is not necessary, except for the historian, to examine it all, for it has been refined, assimilated, and, all but the later developments, superseded in the monumental work of Whitehead and Russell—*Principia Mathematica*—the present culmination of the movement. This work, however, which has not yet been completed, the philosopher must examine *minutely* if he would understand, as a philosopher ought to understand, the fundamental nature of mathematics as disclosed in the best light that has been thrown upon it and especially if he would realize the hope of being able to improve the light, which is not yet perfect. The symbols are at first repellent; they tend to frighten but are not in fact difficult to master.

> They are things of so frightful mien
> That to be hated need only be seen.
> But often seen, familiar with their face,
> We endure them first and then embrace.

Theoretically, the symbols are not essential, a sufficiently powerful god could get along without them; but practically they are indispensable as instruments for economizing our intellectual energy.[7]

No kind of work, whether philosophic or scientific, can be severer in its demands. None surpasses it in respect of the toil involved, nor in patience, nor in depth of penetration, nor in subtlety, nor imagination, nor analytic finesse, nor in the demand it makes upon the *constructive* faculty, and none can give to the competent student a serener vision of eternal things. If on this account it seems to you, as it may seem, a little strange that the majority of mathematicians have little interest in such work and are not familiar with it, it is sufficient to reflect that, though its results as results are strictly scientific, strictly a part of mathematics, they are deeply tinged with philosophic interest and owe their discovery primarily to the spirit of philosophic enquiry. In mathematics, as in other subjects, fashions change; it is, moreover, so large a subject that a student is obliged by his limitations to specialize in a branch of it or in a group of branches; and it so happens that a large majority of mathematicians are disqualified,—some of them by breeding, more of them by temperament,—for study or research in that branch which deals with the foundations of their science as a whole. Such disqualification is not to be imputed to them as a fault; often no doubt,—oftener than not, perhaps,—it is only a defect of a quality; at all events, a mathematician may not be rightly blamed for the temperamental bent of his scientific interests. The same may not be said of those who are inclined to depreciate other interests than their own. I refer to the type of mathematician,—such as you may sometimes meet,—who, as if to mitigate his sense of guilt for being

7. In relation to tbe early history and importance of symbolism do not fail to read Professor David Eugene Smith's beautiful essay, "Ten Great Epochs in the History of Mathematics," in *Scientia,* June, 1921.

consciously innocent of symbolic logic and so to protect his self-respect, will occasionally ask you, in a somewhat disparaging tone, to tell him, if possible, of any important service rendered by symbolic logic or of any important proposition established by it or of any important method devised by it for the use of mathematicians. If you disregard the spirit in which such questions are sometimes asked, it is easy to answer them in a way satisfactory to any candid and competent enquirer. The answer, as I conceive it, is, in brief, as follows:

> (1) Symbolic logic has established the thesis that all existing mathematics (and presumably all potential mathematics) is literally a logical outgrowth of a few primitive ideas, and a few primitive propositions, of logic; and, that, accordingly, logic and mathematics are spiritually one in the sense in which the roots, the trunk and the branches of a tree are physically one: a proposition which, though philosophical and not mathematical, is, in respect of human significance, unsurpassed.
>
> (2) In course of the work establishing the foregoing proposition, symbolic logic has discovered and rigorously demonstrated a long sequence of theorems respecting propositions, classes, and relations, which theorems constitute an immense new body of genuinely mathematical doctrine underlying mathematics as commonly understood and they are open to inspection by all critics, whether friendly or unsympathetic.
>
> (3) Symbolic logic has not promised nor pretended to devise methods to facilitate mathematical research except research in mathematical foundations; in such research the effectiveness of the methods employed is patent in the results.
>
> (4) Finally, symbolic logic is simply the latest fruit of the critical spirit in mathematics—fruit of the refinement,—the inevitable refinement,—of that spirit which has led to so many mathematical developments familiar to all mathematicians,—the postulational method, for example, the birth of non-Euclidean geometries, the theory of manifolds including the hyperspaces, the so-called arithmetization of mathematics, and similar phenomena throughout the history of the science. To depreciate symbolic logic is to oppose the progress of the *spirit* of constructive criticism and that means opposition to the progress

> of science; for Cousin's famous *mot* is just: *La critique est la vie de la science.*

In saying that the philosopher's mathematical obligations require him to familiarize himself with the methods and results of symbolic logic, I have not quite finished the tale. One point remains to be stressed. Before presenting it, let me remind you of a certain fairly obvious distinction which Bergson [8] has emphasized and has elevated, rightly I believe, to the level of an important principle of knowledge. I may best make it clear by an example. You *know,* as we say, how to move your arms. This knowledge is not a part of, and is not derived from, your "scientific" knowledge of physiology, anatomy and physics, though this knowledge, too, may tell you much respecting the motion in question. The latter knowledge is indirect and external—a knowledge from without; the former is immediate and internal—a knowledge from within; it is a living instinct—of the essence of your life; the other is only a superadded understanding. Complete or perfect knowledge of any thing involves both of these kinds of knowledge. In the illustration I have used, the thing to be known is a part of the knower—the mobile arm is yours and its life is yours. But most objects of knowledge are not thus parts of the knower. Of such objects complete knowledge, even if we suppose the element of "understanding" to be perfectible, is unattainable; for to attain it, to gain the other element,—the instinctive element, the inner kind of knowledge,—would require the knower to make the object's life an intimate part of his own; and this, it is plain, cannot be done perfectly. But—and here is a fact of the utmost importance—it can be done approximately. Do you ask, how? The answer is: By the noetic agency of sympathy or love; by the means which Bergson has so finely described as "intellectual sympathy" with the object's life. Your thought, I fancy, runs ahead of my speech and already sees the bearing of the point upon the philosopher's obligations to mathematics. In a sense more than figurative, this science has a life of its own. Else how could it grow? To acquire such knowledge of the science as the philosopher's vocation demands, to know it from within as it instinctively knows itself, he must acquire such intellectual sympathy with it as will enable him to feel its proper life as part of his own. Sympathy so living and intimate,—embracing the instincts, and feeling the

8. "Introduction à la Métaphysique." *Revue de Métaphysique et de Morale,* Vol. II, (1903).

impulses and moods, of an alien life,—is not easily acquired. In the case of mathematics, collegiate courses in algebra, geometry and trigonometry cannot give it, except to the born mathematician, who has it already; neither can it be given adequately by symbolic logic for this study is too meditative for the purpose, too introspective, being more concerned to "understand," than to "live," the life of mathematics. No, if the student of philosophy would acquire that kind of knowledge of mathematics which can come to him only through intellectual sympathy with its life, he must share its life; he must penetrate it deeply enough to feel the touch and thrill, the push and sweep, of its conquering tide; he must at least plunge into Analytical Geometry and the Infinitesimal Calculus where the science first won, and its votaries first win, a worthy sense of its power and its destiny.

In the light of the foregoing considerations, the mathematical obligations of the philosopher appear to be heavy. They are heavy; but they are not too heavy for those whose native talents qualify them for a vocation demanding "magnificence of mind." It is consoling to know that a student who faithfully keeps the obligations will have two great rewards: the joy of an insight and a power not to be otherwise gained; and the joy of representing and perpetuating a noble tradition of his kind,—the tradition, I mean, of mathematical competence as illustrated by the heroes of philosophy in every important age. In relation to that tradition, it is indeed true, as you know, that there have been many philosophers of great learning, some of them important thinkers, whose ignorance of mathematics has been virtually complete, and these have differed widely in kind; of their mathematical ignorance some of them have not been aware; some have deeply regretted it and humbly confessed it—our own beloved William James, for example; in some it has been not only complete but shameless as well, even haughty and defiant, as in Sir William Hamilton and Schopenhauer, whose false and malicious diatribes against mathematics I have dealt with elsewhere,[9] and in case also, I am sorry to say, of Benedetto Croce,[10] whose fine literary and artistic culture and true elevation of spirit have not availed to restrain him from speaking with strange confidence and very disparagingly of a science which his fellow countrymen, by brilliant research, have done so much to honor

9. *Human Worth of Rigorous Thinking,* p. 290.
10. *Logic as the Science of the Pure Concept.*

and which he has not qualified himself to understand even slightly.

It is edifying to compare such representatives of philosophy with its towering heroes, its men of "summit-minds": with Plato, for example, who knew perfectly the mathematics of his time, whose sense and revelation of its spiritual significance has never been surpassed, and whose influence in his own and all succeeding ages has given his name a permanent place in mathematical history; and with Aristotle, whose discussions of such fundamental questions as the nature of mathematical definition, hypothesis, axiom, postulate, and subject matter, are of high value even today and whose great contributions to logic must now be regarded, in the light of modern symbolic logic, as being, though he did not know it, genuine contributions to mathematics; and with Descartes, discoverer of important mathematical propositions, and chief inventor of analytical geometry,—second in scientific power to only one among mathematical methods; and with Leibniz, father of modern symbolic logic and co-inventor with Newton of the infinitesimal calculus, "the most powerful instrument of thought yet devised by the wit of man";[11] and with Spinoza to whose lot it fell to try the great experiment,—inevitable in the history of thought,—of clothing ethical theory,—highest of human interests,—with the strength and beauty of mathematical rigor and form, and, in trying it, to exemplify in a singularly noble way, the fact that illustrious failures fall to the lot of none but illustrious men; and with other great philosophic personalities, if I did not fear to weary you in naming them, who by their mathematical competence worthily represent the heroic tradition.

In closing this initial lecture, I desire to indicate in a general way the sort of topics with which the following lectures will deal. The endless number of the ideas, or notions, or concepts,—as they are variously called,—which enter as components into the stately edifice of mathematics, though they are all of them, in a sense, indispensable to it, yet differ very widely in respect of their place and rank, their dignity and structural service. Examination of the great edifice makes it evident that some of them,—a relatively small number of them,—have the distinction of being related to it as central supporting pillars. Among the chief of these are the concepts denoted by the terms: Function—Propositional Function—Implication—Proposition—Class—Relation—Postulate System—Doctrinal

11. See the preface of Professor W. B. Smith's *Infinitesimal Calculus.*

Function—Doctrine—Variable—Limit—Number—Finitude—Infinity—Transformation—Group—Invariance. It is with such pillar-concepts,—which are obviously not coordinate in rank,—that I purpose to deal, and I shall deal with them primarily as concepts, explaining them with constant regard to clarity, with a minimum of technical symbols, and with a view, not alone to their mathematical meanings, but to their significance and use in outlying fields of thought. But I shall not endeavor to expound, in the proper sense of the term, the great technical doctrines that have grown up about them as subject matter, for such exposition would demand, as you know, not merely one course, but many courses, of lectures. You will rightly infer that, though proof or demonstration may not be entirely absent, it will not be permitted to detain us too long, much less to dominate the discussions.

Let me say, finally, that the course is not designed to be, in the stricter and narrower sense of the term, a course in the philosophy of mathematics. It aims at being at once something less and something more: *less,* in that it does not endeavor to begin with the most ultimate of logical principles and to build upon them, little step by step, with infinite patience, the solid masonry of the mathematical edifice; *more,* in that it is a good deal concerned with the mentioned task of evaluation—with disclosing the relations of mathematics to other great forms of intellectual activity and especially its bearings upon the universal interests of the human spirit.

Lecture II

Postulates

CONCRETE DEFINITION OF POSTULATE SYSTEM—THE PROTOTYPE OF PRINCIPLES OR PLATFORMS—THE ANCIENT "CRAFT OF GEMETRY"—THE SWORD OF THE GADFLY—CLARITY OR SILENCE—MUNICIPAL LAWS AND THE LAWS OF THOUGHT.

The introductory lecture has served, I hope, to indicate in a general way the aim, the spirit, and the scope of our undertaking. In deciding to begin the work proper with a study of the great concept denoted by the familiar term—Postulate System—I have been guided by three considerations: (1) every question arising in what is strictly called the philosophy of mathematics—in the study, that is, of its logical foundations—is connected more or less closely, directly or indirectly, with that concept, which is thus the central ganglion of mathematical philosophy; (2) by means of the concept in question and without unnecessary delay, I desire to set in clear light another concept, intimately related to it, to which I have given the name—Doctrinal Function—and which, if I am not mistaken, has great philosophic importance; (3) postulate systems as employed in mathematics, appear there in perfect light as systems of principles underlying and supporting definite bodies of thought, and so they serve as a model, as an ideal prototype, for the inspiration, the guidance and the criticism of *every* rational enterprise, whether of philosophy, of science, or of life in general.

A subject so fundamental, many-sided, and far-reaching will naturally detain us for some time. The wisdom we seek is golden, but it cannot be gained by any of the get-rich-quick methods characteristic of our industrial and neurasthenic age; the way to it is a little long and I may as well warn you that in these lectures I intend to pursue it in a leisurely fashion. The study is not so “entertaining” as a “movie” nor so easy as the life of “maggots in a cheese” or that of summer birds in a valley of fruits. It demands some patience, hard work and endurance. It will quickly weary such as are content with a little phraseological facility in matters they do not understand, but not those whose curiosity is deep and genuine, for they will be sustained by the dignity of the task and the joy of the game.

Let us now enter upon it. What are we to understand by the term postulate? You are aware that a branch of mathematics (or, for that matter, of mechanics or of physics or of any other science), if the branch be ideally constructed, is autonomous: it consists, that is, of a body of propositions of which a few are assumed—not proved in the branch but taken for granted there—and the rest are deduced from them as logical consequences. To students of philosophy, I need not say that to suppose *all* the propositions of an autonomous theory to be proved in it, plainly involves circularity and a contradiction in terms. In accordance with current usage, which I intend to follow in this matter, any proposition thus taken for granted in a given branch is called a postulate, or assumption, or axiom, or primitive proposition, or fundamental hypothesis, of the branch; these terms being used interchangeably according to the taste of the author. It has not always been so; the term axiom, for example, was long used to denote “self-evident proposition,” which is a kind of proposition that modern mathematicians have not been able to discover. But I shall not detain you with an historical account of the terms, interesting and instructive as their history is. It gives me pleasure to say, however, that, if you feel drawn thereto, as I hope you do, you will find much more than an ample clue to it in the introduction to Dr. T. L. Heath’s superb edition of Euclid’s *Elements* where these terms and kindred matters are set in the bright light of critical commentary from the days of Plato down to the present time. In passing, let me add, by way of indicating an opportunity, that this work of Dr. Heath, like other works of his, attains a high degree of excellence in a type of activity in which our American mathematical scholarship has been singularly lacking; not because American mathematicians have lacked facilities or ability, for these they have not lacked, but because the universities in which they have received their training and have done their work have not yet acquired the requisite atmosphere and spirit.

A postulate is one thing; a system of postulates is another. In defining the former, I have by no means defined the latter. It is not easy to do so with logical precision: it is, I mean, not easy to give an *abstract* definition of the generic concept denoted by the term, postulate system; and I shall not attempt it at this point, for it presupposes study of the concept as actually revealed in mathematics and so has its proper place at the end of the study. Here, at the beginning, we must be content with definition by example, with what Professor Enriques, in his *Problems of Science,* has called *concrete* definition, which is nothing more mysterious than the practice, familiar alike in science and in ordinary life, of telling the meaning of a general term by pointing out one or more of the many objects imperfectly representing it, and saying, "there, there, that is what it means, or that and that." I wish it were practicable in this course to deal adequately with Definition as a separate topic, with its varieties, its functions and its history. It is, I think, an admirable subject for a scholarly dissertation. In such an undertaking the student would find many helpful suggestions in the treatment of definition by Enriques in the work just now mentioned; in certain passages of *Science and Hypothesis* by Poincaré; in some remarkably keen observations found in Pascal's immortal essays, *De L'Esprit Géometrique,* which I cited in the preceding lecture; in the above-mentioned work of Dr. Heath; in the literature of symbolic logic; and, as I need not say to you, who are students of philosophy, in the Metaphysics and Posterior Analytics of Aristotle, not to mention the Platonic Dialogues where philosophy in our western-world first becomes fully conscious that the way to wisdom—to knowledge of things eternal—is not the way of song, however glorious, nor that of sophistry, however pretentious, but is the way of logic, and where accordingly, despite the presence there of many mystical elements, the spirit of Definition, which is the spirit of clear thinking and determinate speech, becomes in Socrates a conquering sword. And this leads me to say, in passing, that in these our democratic times of free speech when everyone, no matter how ignorant or foolish, is a licensed prophet, and blatant sophists abound on every hand, there is no way in which you as teachers of philosophy can render greater service than by carrying on the work of the great Gadfly—constraining men by relentless logical criticism to a choice of one or the other of two alternatives: coherency and clarity of speech or—silence. Today, the mind of the world is a weltering sea of wild passions and wilder opinions. It can not be calmed by municipal law, but it can be by disciplining men to a decent respect for the eternal laws of thought. And that is the supreme obligation of philosophy as the guardian of Reason.

A few moments ago I said that, in the beginning of the study of postulate systems, we must be content to define the notion concretely—by means, that is, of examples. Accordingly, I am going to spread before you presently a definite system of postulates and invite you to examine it as a geologist might examine a specific rock formation; or as a student of poetry might examine a specific poem; or a student of law, the constitution of the Soviet republic or that of the United States. From the large variety of postulate systems recently invented for various mathematical branches, I have selected, as a specimen for our initial study, the system devised by Professor Hilbert and found in his famous *Foundations of Geometry.* It is one of several systems invented in our time to serve as logical bases of Euclidean Geometry. Though it is not intrinsically superior to its rivals, whether in geometry or in other branches, I have selected it in preference to them for two reasons. One of them is that, *practical* arithmetic not being a science, Euclidean Geometry is the oldest and most familiar branch of mathematics, as well as being historically the most interesting and even romantic.

> "The clerk Euclide on this wyse hit fonde
> Thys craft of gemetry yn Egypte londe.
> In Egypte he tawghte hyt ful wyde,
> In dyvers londe on every syde.
> Mony erys afterwarde y understonde
> Yer that the craft com ynto thys londe.
> Thys craft com into England, as y you say,
> Yn tyme of good Kyng Adelstone's day."

From which we see that even in the old island home of our beautiful English tongue the Greek "Craft of Gemetry" has been known for a thousand years. The second reason for my selecting Hilbert's system is that it is the most famous of all existing postulate systems, save one only—that of Euclid. Hilbert's acquired its great fame immediately, not entirely by its merits, for these, as already said, are not superior to the merits of some other systems, but largely through the fame of its author, which was world-wide. If you ask why I have chosen it instead of Euclid's system, which surpasses all others in fame, the answer is that, though

Euclid's system was good enough to withstand more than two thousand years of criticism, it is now known, as we shall see later, to have some grave imperfections—most of them sins of omission. The postulates of Hilbert's system are called axioms by him—"axioms of geometry." As, however, the term axiom as employed by him is exactly equivalent to the term postulate as I have defined it, I shall be doing him no injustice in uniformly referring to his system as a system of *postulates,* thus avoiding the term axiom as likely to suggest the unavailable notion (so-called) of "self-evident truth." The postulates of Hilbert fall into six sets: postulates of connection; of order; of parallels; of congruence; of continuity; of completeness. I give them as found in the authorized English translation of Hilbert's book by Professor Townsend. Physically, the book, as you observe, is small and light; but spiritually it is big and weighty. Except for some harmless abbreviations of statement, the postulates together with the definition of certain terms occurring in them are as follows:

Postulates of Connection

(1) Two distinct points determine a straight line.

(2) Any two points of a straight line determine it.

(3) Three non-collinear points determine a plane.

(4) Any three non-collinear points of a plane determine it.

(5) If two points of a line are in a plane, every point of the line is in the plane.

(6) If two planes have one common point, they have another.

(7) Every straight line contains at least two points; every plane at least three non-collinear points; and space at least four points not lying in a plane.

Postulates of Order

(8) If A, B, C are points of a straight line and B is between A and C, then B is between C and A.

(9) If A and C are two points of a straight line, there is a point B between A and C, and a point D such that C is between A and D.

(10) Of any three collinear points, one, and but one, is between the other two.

(11) Any four collinear points, A, B, C, D, can be so arranged that B shall be between A and C and between A and D, and that C shall be between A and D and between B and D.

Definitions.—A pair of points, A and B, on a line, is a *segment AB* or *BA*; A and B are the *segment's ends*; the points between A and B are the *segment's points.*

(12) Let A, B, C be three non-collinear points and let a be a line of their plane but not containing any of them. If a contains a point of segment AB, it contains a point of segment BC or of segment AC.

Postulate of Parallels

(13) If a straight line a and a point A, not in a, be in a plane α, there is in α one and only one straight line containing A but no point of a.

Postulates of Congruence

(14) If A and B are two points on a straight line a, and if a point A' be on a straight line a', then on either side of A' there is one and but one point B' such that the segment AB is congruent to the segment $A'B'$. Every segment is congruent to itself.

(15) If a segment AB is congruent to a segment $A'B'$ and to a segment $A''B''$, then $A'B'$ is congruent to $A''B''$.

(16) If segments AB and BC of a straight line a have no common point but B, and if segments $A'B'$ and $B'C'$ of a straight line a' have no common point but B', then, if AB and BC are respectively congruent to $A'B'$ and $B'C'$, AC is congruent to $A''C''$.

Definitions.—If O be a point of a straight line a, the points of a on a same side of O constitute a *half-ray emanating from O*; a pair of half-rays, h and k, emanating from a point O and not being parts of a same straight line is an *angle* (h, k); O is the angle's *vertex,* and h and k its *sides*; its *interior* is the class of points such that, if A and B be any two of them, segment AB contains no point of h or k; its *exterior* is composed of all other points of the plane except O and the points of h and k.

(17) Given an angle (h, k), a line a' in a plane α; a point O of a'; and in α a half-ray h' emanating from O; then in α and emanating from O there is one and but one half-ray k' such that the angle (h', k') is congruent to (h, k) and that the interior of (h', k') is on a given side of a'.

(18) If the angle (h, k) is congruent to (h', k') and to (h'', k''), then (h', k') is congruent to (h'', k'').

(19) If, in the triangles ABC and $A'B'C'$, AB, AC and angle BAC are respectively congruent to $A'B'$, $A'C'$ and angle $B'A'C'$, then the angles ABC and ACB are respectively congruent to the angle $A'B'C'$ and $A'C'B'$.

The Postulate of Continuity (or of Archimedes)

(20) Let the point A_1 be between any two given points A and B of a straight line a. Let the points A_2, A_3, A_4, … of a be such that A_1 is between A and A_2, A_2 is between A_1 and A_3, and so on, and that the segments AA_1, A_1A_2, A_2A_3, … are mutually congruent. Then in the point series there is a point A_n such that B is between A and A_n.

Postulate of Completeness

(21) To a system of points, lines and planes it is not possible to add other elements such that the system thus generalized shall form a new geometry in which all the postulates of the foregoing five sets are valid.

Such is a list of the postulates devised by Hilbert to serve as a foundation of Euclidean geometry. I regret having had to detain you so long in the rather arid business of presenting so long a list in detail. My apology is the importance of having the list definitely before us. In closing this lecture, let me recommend that, as a preparation for the next one, you familiarize yourselves with the postulates and in doing so, that you read enough of Hilbert's book to see how carefully the theorems are deduced from the postulates and how inevitably they follow therefrom.

Lecture III

Basic Concepts

PROPOSITIONAL FUNCTION AND DOCTRINAL FUNCTION—MARRIAGE OF MATTER AND FORM—ITS INFINITE FERTILITY—PROPOSITIONS AND DOCTRINES THE OFFSPRING—VERIFIERS AND FALSIFIERS—SIGNIFICANCE AND NON-SENSE—A QUESTION ASKED BY MANY AND ANSWERED BY NONE.

All postulate systems have certain properties or features in common. In connection with the Hilbert system, I desire to draw your attention to such of these features as will lead us to form a certain conception which I think highly important and to which I have given the name—Doctrinal Function.

As a preliminary, I must explain briefly a closely related term—Propositional Function—invented by Bertrand Russell; it is, perhaps, the weightiest term that has entered the nomenclature of logic, or mathematics, in the course of a hundred years. It has the rare distinction of being, as we shall see, a perfect name for a supreme concept. Every one is familiar with the *ordinary* notion of a function—with the notion, that is, of the lawful dependence of one or more variable things upon other variable things, as the area of a rectangle upon the lengths of its sides, as the distance traveled upon the rate of going, as the volume of a gas upon temperature and pressure, as the prosperity of a throat specialist upon the moisture of the climate, as the attraction of material particles upon their distance asunder, as prohibitionary zeal upon intellectual distinction and moral elevation, as rate of chemical change upon the amount or the mass of the substance involved, as the turbulence of labor upon the lust of capital, and so on and on without end. This familiar notion of mutual dependence and mutual variation thus exemplified in every turn and feature of life and the world, is indeed a powerful concept; it is, in a sense, the sole subject matter of science; its scientific name—function—was first pronounced, it is said, by Leibniz; in modern mathematical analysis, it has played a dominant rôle, giving both name and character to certain great branches, as the theory of functions of real variables and the theory of functions of complex variables. Yet, powerful as it is, this Leibnizian conception, as employed in traditional mathematics, is far inferior in scope to that denoted by propositional function, which indeed embraces the former as a special case. What, then, are we to understand by this great term?

The answer, describing rather than strictly defining, is that a propositional function is any statement containing one or more real variables, where, by a real variable, is meant a name or other symbol whose meaning, or value as we say, is undetermined in the statement but to which we can at will assign in any order we please one or more values, or meanings, now one and now another. I fear that what I have just said is too general to be quite intelligible. The idea can be made sufficiently clear, however, by some simple examples—by concrete definition—provided you will understand that the examples are to the general concept in question as a burning match to a world-conflagration or as a few water drops to a boundless ocean. If we denote the real variables by such symbols as x, y, z, w, etc., then for simple examples of what is meant by propositional function we may cite the following quite at random: x is a man; x is a lover of y; x is the specific gravity of y; x is a noble citizen intemperately desiring to impose abstinence on y; x has been divinely appointed by y to subjugate z; $2x - 3y = 10z + w$; $\sin x = \cos y$; x denied that y said that z confessed to being the author of w; x knows that y voted against z on account of jealousy of w; and so on *ad infinitum.* How many variables may enter a propositional function? As many as we please. How many such functions are there? Their name is legion—the host of them is literally infinite. Even so, you may wish to say, the examples are not impressive. Nevertheless, the concept they represent, each in its little way, is sovereign—"like Jupiter among the Roman gods, first without a second." [1] Its majesty, its power, its subtlety, the immeasurable depth and range of its significance can not be perceived and felt at once, but only

1. Gladstone.

more and more with days and months and years of reflection. You will reflect upon it a very great deal if ever you enter seriously upon the study of symbolic logic.

Let us reflect a little upon it now. There will be occasion to resume its consideration at a later stage. At present, I wish merely to direct your attention to the very significant fact that propositional functions, though they have the *forms* of propositions, are *not* propositions. It is of the utmost importance to bear that in mind. A proposition is a statement that is true or else false. That is why propositions are so important—they, and not human hearts, are the residences—the dwelling places—of those curious things called Truth and Falsehood. A propositional function, owing to the presence in it of variables, is neither true nor false. The statements $2 + 7 = 9$, $3 + 7 = 9$, are propositions, one of them true, the other one false; but the statement, $x + y = 9$, is neither true nor false; it is not a proposition but is a propositional function.

You see at once that to derive propositions from a propositional function it is *necessary* to replace the latter's variables with what we may call constants, or values—with terms of definite meaning; but such substitution, though necessary, is not sufficient, for it is always possible to substitute such constants as will give, not a proposition, but nonsense. Suppose, for example, that our given function is the statement, x is an integer less than 5. Now, the *class* of all integers less than 5 is a constant—a definite somewhat. Substituting it for the variable x, we get the statement, the *class* of all integers less than 5 is an integer less than 5. This statement is neither a propositional function nor a proposition; it is nonsense—nonsense consisting in talking of a class of things as if a given class could conceivably be one of the things composing it; as if the class, for example, of locomotives were itself a locomotive; or as if the class of prohibitionary moralists were itself a holy constituent thereof; or as if the class of apples or of asses were itself an apple or an ass. Such "talking" is sheer chattering, as if there were no such things as laws of Thought. It is evident that a propositional function is a *matrix* of the propositions derivable from it by substitution and has the same *form* as the propositions it thus moulds. This latter fact should be noted carefully for in logic—that is to say, in mathematics—form is all-important—so important indeed that some critical thinkers have ventured to call mathematics the science of Form.

The constants that convert a given propositional function into nonsense may be called *inadmissible* constants for that function; all other constants may be called *admissible* constants for the function since they convert it into propositions. It is worthy of note, in passing, that the line of cleavage between the admissible and the inadmissible constants for a given function is not always sharply defined. You can readily construct or find functions of x in respect of which it may be doubtful whether certain constants—the sweetness of sugar, for example, or the glory of renown—are admissible or not. You stand here before an open and inviting field for research, the problem being to determine criteria for deciding, in the case of any propositional function, what constants in the universe of constants are admissible and what ones are not. The situation may be likened to that of physical organisms, for there are plants and there are animals, but in the case of some living organisms there is at present no means of deciding to which division of the kingdom they belong.

The admissible constants for a given function fall into two classes: those converting it into true propositions and those converting it into false ones. It is convenient to call the constants of the former class *verifiers* of the function; and those of the latter class *falsifiers* of it. The verifiers of a function are said to *satisfy* it and are called the *values* of its *variables*; and the propositions derived from a function by substituting values of its variables for these are called *values* of the *function*. Thus, you see that a propositional function is itself a variable—albeit of a different type from the variables it contains—having for its values the true propositions derivable from it by means of its verifiers.

With the foregoing ideas and distinctions in mind, let us return to the Hilbert postulates and ask: Are they propositions or propositional functions? To answer, it is necessary and sufficient to ascertain whether or not they contain variables. We observe at once the presence in them of certain *substantive* terms—"point," "straight line," "plane," and "space"—which seem to denote the things about which the postulates talk, their subject-matter—and certain *relational* terms—"between" and "congruent"—which have the air of denoting definite fundamental relations among the "points" or figures composed of them. We must now ask: Do these terms denote constants—things of unique and definite meaning—or do they play the rôle of variables? Euclid does indeed, as you know, give what he calls "definitions" of point, line and plane, but in his proofs and constructions he makes no use whatever of the so-called definitions, which he ought to have called *descriptions* designed merely to indicate what *he* meant by the terms; or, better, he ought to have omitted the definitions as *logically* useless. As to the term, space, it does not, as it should not, occur in Euclid's *Elements.* By examining Hilbert's book, you will find that he does not attempt either to define or to describe any of the above-mentioned six terms, except, of course, in so far as they are defined—restricted in their possible meanings—by having to satisfy, or verify, the postulates. The omission of all other definition of them is deliberate. And so our question is reduced to this: Does the requirement that the things denoted by the six terms—"point," "straight line," etc.—make the terms constants, assign to each of them a unique and definite meaning? The answer is No: each of the terms admits of many, infinitely many, different definite meanings satisfying the postulates. The answer will be justified at a later stage of our discussion. For the present, I ask you to assume its correctness. We may, therefore, now state, in answer to our main question, that the six terms are not constants, but variables, and that, accordingly, the postulates are not propositions, as they are wont to be called, but are propositional functions. As you reflect upon this fact, you will find that its importance is immeasurable, not only for philosophy in its narrower sense, but for Criticism[2] in the widest sense, in all its fields. In a future lecture, I shall return to the matter of estimating the fact's general importance. For the present, let us follow its strictly logical and philosophical leading.

We have to say at once that the postulates of the system we are examining as a representative specimen of postulate systems in general, are neither true nor false, being propositional functions. The same must, of course, be said of all the theorems deduced or deducible from them as their logical consequences or implicates, for all such theorems, being statements involving the same variables as are present in the postulates, are propositional functions and are, therefore, neither true nor false. At this point, I cannot refrain from pausing long enough to point out how the most vitally fundamental fact in logical theory appears here with startling vividness in new light. Suppose that in the postulates we replace the seven terms—"point," "straight line," "plane," etc.—respectively, by any meaningless vocables whatever, as *loig, boig, ploig,* etc., so that pos-

2. In this connection the reader should consult Professor F. C. S. Schiller's very suggestive article "Doctrinal Functions" in *The Journal of Philosophy, Psychology and Scientific Methods, Vol. XVI., 1919.*

tulates (1) and (3), for example, shall read: (1) Two distinct loigs determine a boig; (3) Any three loigs not in a same boig determine a ploig. Imagine the other postulates to be similarly restated. Then, of course, all the theorems and indeed the entire Hilbert book will discourse explicitly about loigs, boigs, ploigs, etc., and nothing else. Do not fail to note now, once for all, that as thus restated, the theorems and postulates are *related* precisely as before—the former being logical consequences of the latter and deducible therefrom without even the slightest change in the reasoning. The fact which thus leaps naked into view is that logical deduction,—mathematical demonstration,—*all* valid proof in no matter what subject-matter,—depends *entirely* upon the *form* of the premises, or postulates, and not at all upon any specific meanings we may assign to their undefined, or variable, terms or symbols. What is meant by propositional *form?* The question has been often asked but never answered. I ask it here merely to signalize its importance. It is exceedingly difficult. I hope we may return to it later. At present, let us go on with the central thread of this lecture.

We have seen that the Hilbert postulates and all the theorems logically deducible from them are propositional functions. So important a fact ought not to be concealed, not even from the physical eye. To lay it bare, it is sufficient to replace in the postulates the terms, there playing in disguise the rôle of variables, with proper symbols for variables; substituting, let us say, the symbols v_1, v_2, v_3, v_4, respectively, for the substantive, or element-naming, terms,—"point," "straight line," "plane," "space,"—and for the relational terms,—"between," "congruent,"—the symbols R_1 and R_2. Then postulate (1) will read: Two distinct v_1's determine a v_2. For another example, postulate (8) will read: If v_1', v_1'', v_1''' are v_1's of a v_2 and v_1'' has the relation R_1 to v_1' and v_1''' then v_1'' has R_1 to v_1''' and v_1'. It is obvious that all of the postulates and theorems admit of such restatement. I strongly recommend that, as a very enlightening exercise, you thus restate all of the postulates, a few of the theorems, and rewrite the proof of at least one of the latter.

Having thus dragged into solar light the fact,—hitherto evident only in the psychic light of understanding,—that our postulates and theorems involve variables, let us now think of the postulates and theorems as constituting a Whole—a definite Body of logically related propositional functions. Not one of them is true; not one of them is false. What is true is that *the postulates imply the theorems.* But this statement of implication, though it is a proposition and is a true one—is not a part of the Whole; it is not contained in the Body of functions; were we to put it in, it would stand there alone as an intruder, being neither one of the postulates nor one of the theorems, neither a premise nor a conclusion, neither an implier nor an implied; it is a philosophical proposition *about* the Whole but is not a member of it; it is a critical commentary upon it but not upon itself; it is a judgment,—a just and important judgment,—regarding the Body of propositional functions, but is wholly external to it.

This definite Body of logically compendent propositional functions, if one will but meditate upon it, is a truly wonderful thing—a great indestructible shining Form of forms—"poised in eternal calm" above the changeful things of the world of sense. What shall we call it? It is evidently one of many, for every postulate system gives rise to such a Form and many of these systems, as we shall see, are essentially different. Shall we call it Euclidean Geometry as Hilbert called it with the world's consent? A part of our future task is to show that it has neither more nor less to do with geometry as this term has been understood from time immemorial than with a thousand other things. Shall we say it is a Doctrine of a certain kind? No; for a doctrine must have a specific subject-matter, which our Form has not; it must consist of propositions, which our Form does not; it must be true or else false, but our Form is neither.

What, then, shall we say it is? What, pray, ought our Form,—our definite autonomous Body of propositional functions,—to be called? Observe that if we replace the variables in its postulated functions by admissible constants, we thus obtain a body of propositions matching, in one-to-one fashion, *all* the functions of our Body of functions; we thus obtain, that is, a *doctrine,* for the body of propositions has a specific subject-matter and is true or false according as the substituted constants are all of them verifiers, or some of them falsifiers, of the postulated functions. Obviously, we may thus obtain various doctrines from our Body of functions by substituting various sets of admissible constants for the variables in the postulated functions. It is obviously natural to call the true doctrines thus derivable the values of the Body of functions.

It is now as plain as the noon-day sun what the answer to our question must be: our Body of logically related propositional functions, since it is a thing having doctrines for its values must be named a Doctrinal Function. The same name must, of course, apply to the function body consisting of the postulates of any other postulate system together with the theorems logically deducible from them. It can hardly escape your attention that just as a propositional function has true propositions for its values, a doctrinal function has true doctrines for its values; that just as we viewed a propositional function as the matrix of all the propositions (true or false) derivable from it by substitution of admissible constants, so we may view a doctrinal function as the matrix of all the doctrines (true or false) derivable from it in like manner; and that just as a given propositional function and the propositions derivable from it are identical in form, so a given doctrinal function and the doctrines derivable from it are the same in respect of form; they are *isomorphic,* as we say. In marriage with subject-matter, a Doctrinal Function becomes the matrix of an infinite family of doctrines; the children inherit the form of the mother.

It will be convenient to say that we are *interpreting* a given doctrinal function whenever we derive from it, in the way now familiar, one of its values, or true doctrines; and these values, or true doctrines, may be conveniently called *interpretations* of the function.

Lecture IV

Doctrinal Interpretations

A MOTHER OF DOCTRINES MISTAKEN FOR HER ELDEST CHILD—INFINITELY MANY INTERPRETATIONS OF ONE DOCTRINAL FUNCTION—ORDINARY GEOMETRY BUT ONE OF THEM—OTHER INTERPRETATIONS GEOMETRIC, ALGEBRAIC AND MIXED—IDENTITY OF FORM WITH DIVERSITY OF CONTENT—DISTINCTION OF LOGICAL AND PSYCHOLOGICAL—PROJECTIVE GEOMETRY THE CHILD OF ARCHITECTURE—A SCIENCE BORN OF AN ART—INFINITE POINTS AND THE MEETING OF PARALLELS—POLE-TO-POLAR TRANSFORMATIONS—LOGICAL USE OF PATHOLOGICAL CONFIGURATIONS.

In the following discussion, I shall assume that you have before you the Hilbert postulates as restated in terms of the variable-symbols, v_1, v_2, v_3, v_4, R_1 and R_2. It will be convenient to call the doctrinal function consisting of these postulates and their consequent theorems the "Hilbert doctrinal function" and to denote it by $H\Delta F'$. Now be good enough to note very carefully that, if we omit from the postulates all reference to points not in a given plane, the remaining postulates together with their theorematic consequences constitute another doctrinal function and that this is included in $H\Delta F'$. Let us denote the minor function by $H\Delta F$. The purpose of this lecture is to present or rather to indicate some of the infinitely many values, or interpretations, of these two functions; to indicate, that is, some of the true doctrines having the functions for their common mould.

One of the interpretations of $H\Delta F'$ is the familiar doctrine which results from letting the symbols, v_1, v_2, v_3, v_4, R_1, R_2, denote, respectively, *point, straight line, plane, space, between* and *congruent,* or *equal,* taken in the sense in which they have been taken from pre-Euclidean days,—in the sense in which they (or some of them) are "described" by Euclid in the *Elements,*—in the sense in which Hilbert takes them in his *Foundations* as shown by the drawings or figures he there employs and which is doubtless responsible for his calling his book *The Foundations of Geometry.* This special interpretation of $H\Delta F'$,—this special value of that function,—this special doctrine, which I shall denote by D_1',—is, you observe, the ordinary Euclidean Solid Geometry, or geometry of three dimensions, with which we all of us gained some acquaintance in high school or college despite the somewhat rough or uncritical way in which it was there presented as for beginners. The corresponding interpretation of $H\Delta F$ is the yet more familiar Euclidean geometry of the plane, a two-dimensional geometry. Denote it by D_1. I shall take both D_1' and D_1 for granted, assuming them, whenever it is convenient to do so, in future discussion.

Let me now direct your attention to another geometric interpretation of the two functions—to one which, though it is near-lying and fairly obvious, has not, so far as I am informed, been published. In order to present it intelligibly, I must, by way of preparation, make you acquainted with the concepts of projective straight line, projective plane and projective space, for, as you will recall, I have not assumed on your part a knowledge of Projective Geometry. It will be sufficient for our purpose to introduce them in the rough traditional way instead of the very refined way employed by Veblen and Young, for example, in their Projective Geometry, which is based upon a postulate system appropriate for projective geometry.

Let the figure be in a Euclidean plane—the kind of plane belonging to D_1. All lines of the plane that contain a given point P constitute a *pencil* of lines; P is the pencil's *vertex.* All the points of a line L constitute a *range* of points; L is the range's *base.* It is plain that each point of range L is on one line of pencil P; and that, reciprocally, each line of P has one point of L, with a single exception,—L', parallel to L, contains no point of L. To remove this exception to the one-to-one correspondence, otherwise perfect, there is made in projective geometry an agreement or convention: namely, that each line has (at an infinite distance) a so-called "ideal" point, or point at infinity, and that the "ideal" points of any two parallel lines are coincident. We thus get, as you see, a new sort of straight line and of plane and of space, which we describe by calling them respectively

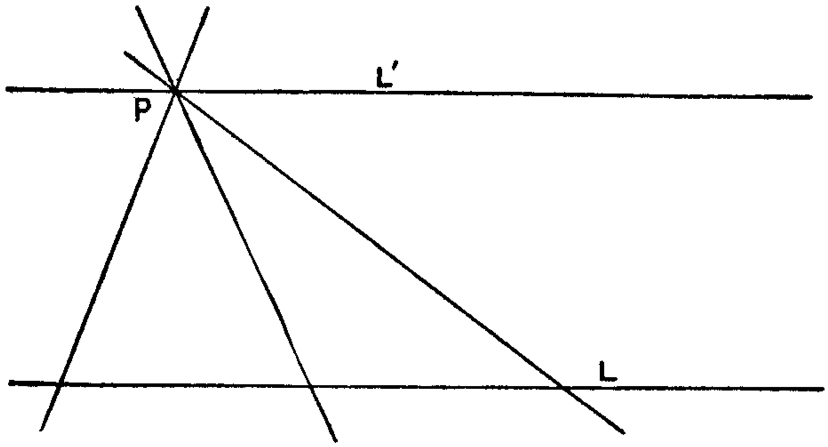

FIG. 1.

projective straight line, projective plane and projective space. The adjective has fine propriety, but that need not here detain us. You can readily prove, or you may assume, that the "ideal" points of the projective plane constitute a straight line—called the "ideal" line, or line at infinity; and that the locus of the "ideal" points of projective space is a plane—called the "ideal" plane, or plane at infinity. I can not pause here to justify the convention. It is amply justified by its consequences, for which, if you be interested, you must repair to projective geometry,—invented by the engineer, Desargues, a contemporary of Descartes and Pascal,—quickly forgotten—reinvented, in France again, about one hundred years ago—perhaps the most beautiful branch of mathematics.

We may now proceed to the promised new interpretation of our doctrinal functions. As $H\Delta F$ is simpler than $H\Delta F'$, let us first deal with the former.

Let π denote a projective plane. Let a chosen point O be the vertex of a pencil of lines of π; call each line of the pencil an O-line. Note that every other pencil of π contains one and but one O-line. Now let us in thought remove from π, once for all, the O-pencil. We thus remove one and but one line from every other pencil. We may conveniently call the pencils, thus bereft of a line, *pathopencils* as being defective or, so to speak, pathological. We have taken from π one and but one pencil of lines. Our field of operation consists of all that is left. Denote the field by Φ. We are going to give $H\Delta F$ an interpretation in Φ; the interpretation, as

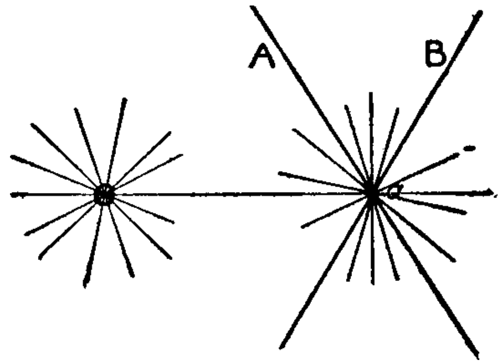

FIG. 2.

you will see, will be a doctrine about certain things in Φ—a geometry of the field. The interpretation results from assigning to the v's and the R's in the postulates of $H\Delta F$ the following meanings, or constant values: v_1 is to mean a line of Φ; v_2, a pathopencil of Φ; R_1 is to mean "between" in the sense that, if A, B, C be three lines of a pathopencil, B will be considered to be between A and C, if A (or C) must rotate through the position of B to coincide with C (or A) (for, of course, a line of a pathopencil must not be supposed to rotate into the position left vacant by the absent O-line); and R_2 is to mean "congruent" in a sense to be given later.

We have to show that the indicated meanings satisfy, or verify, the postulates of $H\Delta F$. That some of them are thus satisfied may be made evident by simple figures; and it will be interesting and enlightening to exhibit such evidence before giving the proof for all the postulates. At the same time, we will lay bare, by means of figures, the significance of one or two theorems of the new doctrine. I shall not here repeat the postulates, but will suppose you to have them in hand.

Postulate (1) is plainly satisfied, for any two lines A and B of Φ determine, as in Fig. 2, a pathopencil a, which consists of all the lines through a except the O-line Oa.

Next consider postulate (8). That it is verified is evident in Fig. 3 where line B is clearly between A and C and between C and A. For another example, let us take postulate (12), the famous postulate of Pasch. But first we must have some

DEFINITIONS.—A pair of lines, A and B, of a pathopencil, is a *segment* AB or BA; its *ends* are A and B; the lines between them are the *segment's*

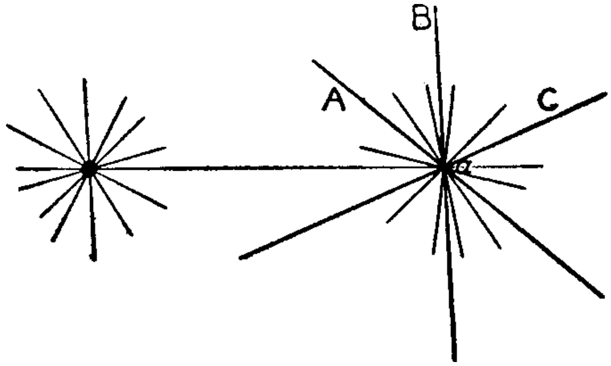

FIG. 3.

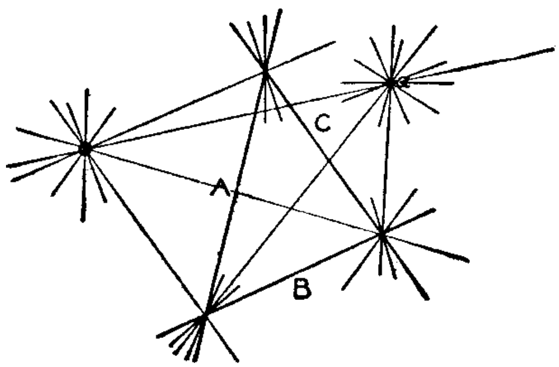

FIG. 4.

lines.

In the light of Fig. 4, it is obvious that postulate (12) is satisfied. Note that *A*, *B*, *C* are any three lines of Φ not belonging to a same pathopencil; that pathopencil *a* contains a line of segment *AB*, by hypothesis; and that *a* contains a line of segment *BC* but none of segment *AC*.

Before considering another postulate, let us illustrate the following *theorem* (a propositional function in the doctrinal function *H*Δ*F*): *Any given* v_2 *separates the remaining* v_1*'s of the* v_3 *into two classes such that, if* v_1' *and* v_1'' *are one of them in one of the classes and the other in the other, the segment* $v_1'v_1''$ *contains a* v_1 *of the* v_2*; and that, if* v_1' *and* v_1'' *are both in one of the classes, the segment does not contain a* v_1 *of the* v_2. (It is theorem 5 of Hilbert's book.) A fairly careful examination

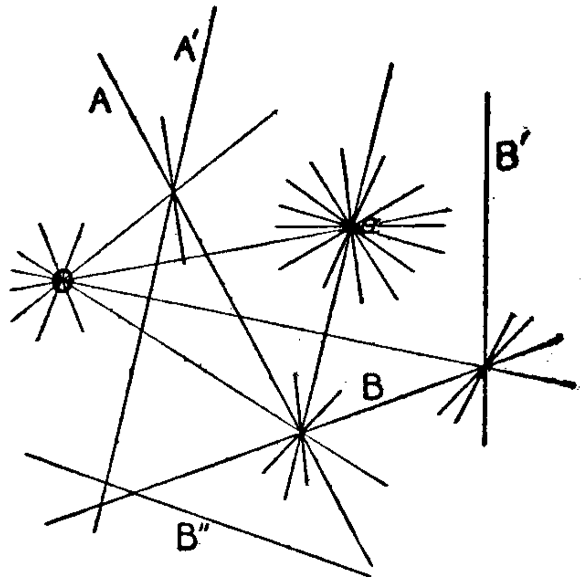

FIG. 5.

of Fig. 5 will suffice to convince you that that theorem is verified in our new interpretation. One of the two classes of lines is composed of all the lines of Φ that go between O and a; all the other lines compose the other class. Segments AA' and BB' contain no line of the pathopencil a, but any such segment as AB contains a line of a.

You should not fail to compare Fig. 5 with Hilbert's figure for the corresponding proposition in doctrine D_1, the old familiar interpretation. The two figures are the same logically but very different psychologically: in the latter figure the truth of the proposition is perfectly and immediately evident to intuition, while in the former the truth of the proposition is very far from being thus evident. Why? The question, you observe, is one for psychologists, like hundreds of similar questions that arise here and elsewhere in mathematics, if only psychologists would learn enough mathematics even to *ask* the questions.

Let us now turn to postulate (13)—the postulate of parallels. Fig. 6 shows clearly that this famous Euclidean postulate is satisfied by our new interpretation. Here a is the given pathopencil; A is any given line not belonging to a; b is a pathopencil containing A but having no line in common with a, and there is plainly no other such pathopencil; in

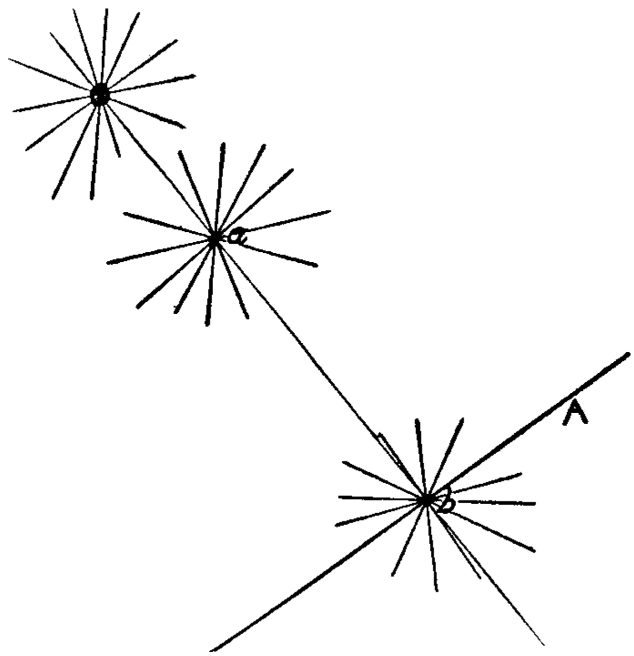

FIG. 6.

other words, b is parallel to a and there is no other such pathopencil containing A.

If, now, you attempt to show (and I advise you to make the attempt) by a figure that postulate (20), or other postulate involving *congruence,* is satisfied, using "congruent" in your figure in the sense it has in the old interpretation or doctrine D_1, you will quickly find yourselves in trouble. In the new interpretation, however, we are not going to employ "congruent" in that sense, but in a sense which I shall explain presently in the course of a simple argument designed to show, as by a single stroke, that *all* of the postulates are satisfied by our new interpretation.

Before presenting that argument we must acquaint ourselves with what is called, in the projective geometry of a plane, the *Pole-Polar transformation with respect to a circle.* It is a very beautiful transformation, important, and easy to understand.

Let Fig. 7 be in a projective plane π. Tangents through P are drawn to the circle. Line L joining the points of tangency is called the *polar* of P, which is called the *pole* of L. You readily see that, if P moves off, becom-

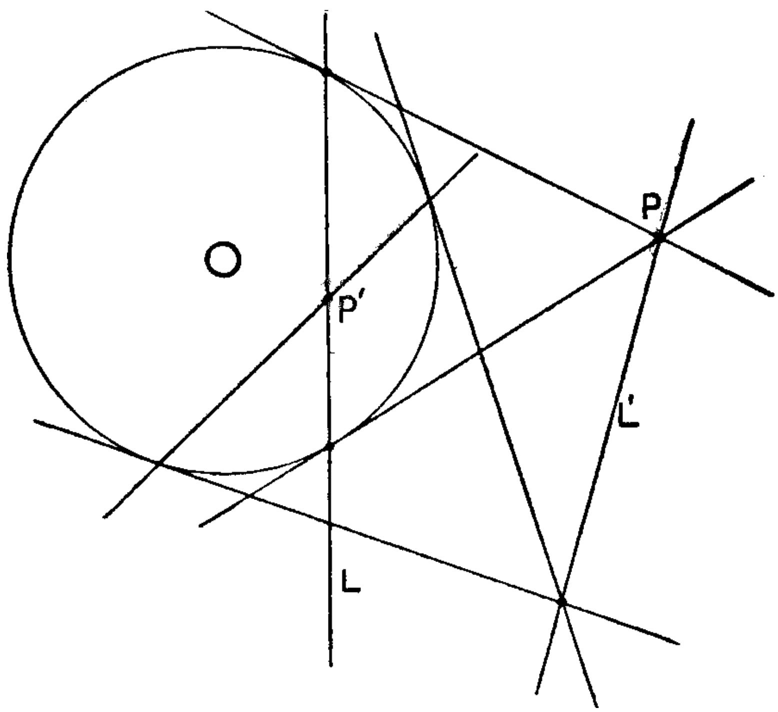

FIG. 7.

ing an “Ideal” point of π, the polar L goes through the center—is a line of the pencil vertexed at O; also, if P moves up to the circle, L becomes tangent at P. If P is inside the circle, say at P', L is L', whose construction is obvious; if, in particular, P is at O, L is π’s “ideal” line, or line at infinity. Thus you see that the given circle serves to set up a one-to-one correspondence between the points of π as poles and the lines of π as polars. This correspondence is called the Pole-Polar transformation of π with respect to the circle. We say the transformation transforms or converts a point into its polar line, a line into its pole point, and each of these is called the *transform* of the other. If you will study the transformation a bit, playing with it, making a few figures, you will discover some of its important properties, such as these: it converts a *range* of points into a *pencil* of lines, and a pencil into a range; a segment of a range into a segment of a pencil, and a pencil segment into a range segment; if three points of a range or three lines of a pencil are in the order—A, B, C,—the transforms are in the same order.

And now for the argument showing that all the postulates in $H\Delta F$ are verified by our new interpretation. Imagine our field Φ laid down upon a Euclidean plane α. Remember that the O-pencil is not in Φ—I have put in a few of its lines merely to remind us that it is absent. Such a pencil is present in α just below. Remember also that Φ has an “ideal” line at infinity which α has not. Assume a definite circle C about O as center. Consider the pole-polar transformation as to C. Let the transforms of the points and lines of α be in Φ; you readily see that, in a one-to-one way, the points of α are converted into the lines of Φ and the lines (ranges) of α into the pathopencils of Φ; also that the *order* of the elements in α is carried over into their transforms in Φ. But, as you readily see, congruence in α,—that is, congruence as understood in interpretation D_1,—is not carried over. We, therefore, agree to give a new meaning to “congruent” for use in Φ, and the meaning is this: if two segments or angles be congruent (in the old sense) in α, then and only then their transforms shall be said to be congruent in Φ. It is evident, without further talk, that all the postulates are satisfied and that we, accordingly, have a new interpretation of the doctrinal function $H\Delta F$. Let us denote this interpretation, or doctrine, by D_2. D_2 is evidently a two-dimensional geometry of the lines and pathopencils of Φ and is isomorphic with D_1, the ordinary geometry of the points and lines of a Euclidean plane.

I will close this lecture by indicating,—merely indicating,—the analogous new interpretation of the Doctrinal function $H\Delta F'$, which, you remember, includes the entire list of Hilbert postulates in their restated form. I shall denote the new doctrine, or interpretation, by D_2'. Let S denote a projective space of three dimensions. We have already formed the concept of such a space. All the lines (or planes) of S that have in common point P are together called a *sheaf,* or *bundle,* of lines (or planes); all the planes having a common line constitute an *axial pencil* of planes. Let O be a chosen point of S. Call the sheaf of lines (or planes) having O for vertex the O-sheaf of lines (or planes). In thought remove from S the O-sheaf of lines and the O-sheaf of planes. We thus remove from every other line sheaf one line, from every other plane sheaf an axial pencil and from every axial pencil (not contained in the O-sheaf of planes) one plane. The ensembles, thus rendered defective, may be respectively called a pathosheaf of lines, a pathosheaf of planes and a pathopencil of

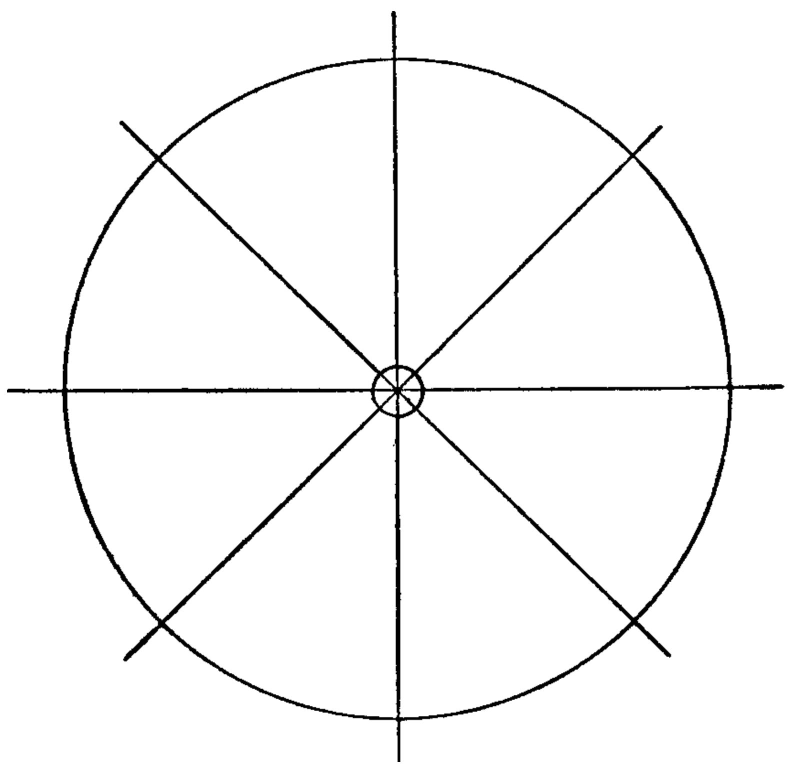

FIG. 8.

planes, or plane pathopencil. Analogous to the pole-polar transformation as to a circle,—which we have already explained and used,—there is for S a pole-polar transformation with respect to any given *sphere* converting each point into a polar plane and each plane into pole point. Our field of operation—Φ'—is S bereft of the two O-sheaves. As you may have by this time surmised, our new interpretation, or doctrine D_2', arises on giving the variable symbols in the postulates of $H\Delta F'$ meanings as follows: v_1 will mean a plane of Φ'; v_2, a pathopencil of planes; v_3, a pathosheaf of planes; v_4, Φ'; R_1, between in the sense that, if A, B, C are planes of a pathopencil, B will be said to be between A and C if either of the latter must rotate through the position of B to coincide with the other; R_2 will mean congruent in the sense that segments, etc., in Φ' will be called congruent if they are transforms of segments, etc., congruent in D_2.

Obviously D_2' is a three-dimensional geometry of planes, pathopencils of planes and pathosheaves of planes of Φ' and is isomorphic with D_1', the familiar geometry of points, lines and planes of ordinary Euclidean space.

Note that D_2 and D_2' are *logically* the same as D_1 and D_1' but greatly differ from the latter *psychologically.*

Lecture V

Another Geometric Interpretation

BRIEF INTRODUCTION TO THE METHOD OF DESCARTES AND FERMAT—INVERSION GEOMETRY AND INVERSION TRANSFORMATION—THE INFINITE REGION OF INVERSION SPACE A POINT—BUNDLES OF CIRCLES AND CLUSTERS OF SPHERES—PATHOCIRCLES AND PATHOSPHERES—ONE-TO-ONE CORRELATION.

In presenting a third interpretation of our two doctrinal functions, it will be convenient to borrow a few ideas from Cartesian Analytical Geometry and Inversion Geometry. It will be advantageous to explain them in advance.

The perpendicular lines OX and OY, Fig. 9, are called *coordinate axes*; O is the *origin* of distances, which, if measured upward or rightward, are regarded *positive,* but, if downward or leftward, *negative.* I am supposing the figure to be in a Euclidean plane. Choose some unit of length; then any point has a pair of numbers (x, y), P's distances from the axes and called its *coordinates.* Conversely, to any such a pair belongs a point. Let (1), Fig. 10, be any line through O; then (2), parallel to (1), is any line of the plane. Let $P(x, y)$ be any point of (1); let $m = \tan\theta$; then $y = mx$; this equation is the equation of (1); it is so called because to any pair (x, y) satisfying it belongs a P of (1) and any P of (1) has a pair satisfying it. Plainly, the y of P' is equal to P's $y + b$; hence the equation of (2), any line of the plane, is: $y = mx + b$. Conversely, any equation of first degree in x and y represents a line of the plane.

By Fig. 11 you see that, if d is the distance between two points, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$; then $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$.

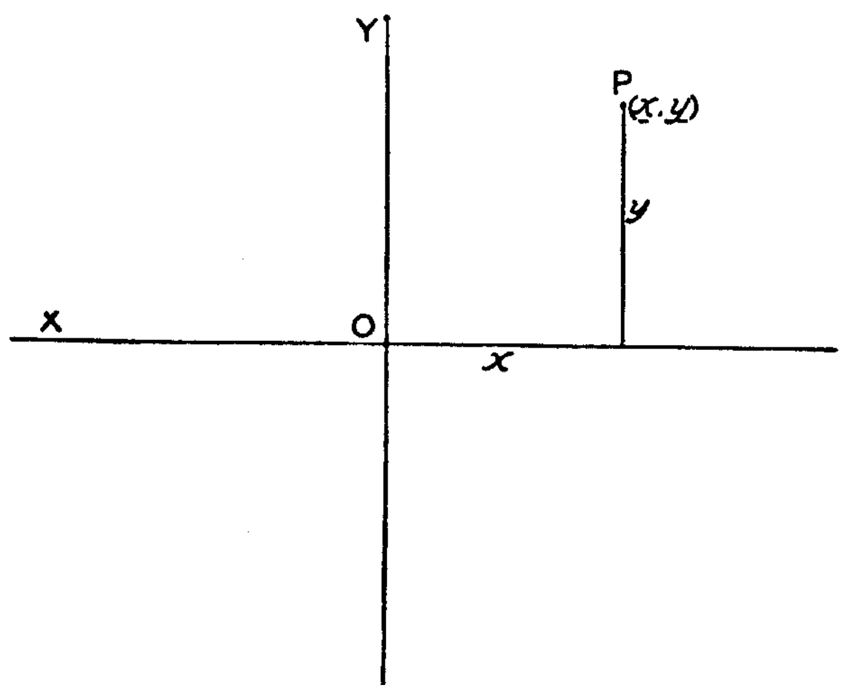

FIG. 9.

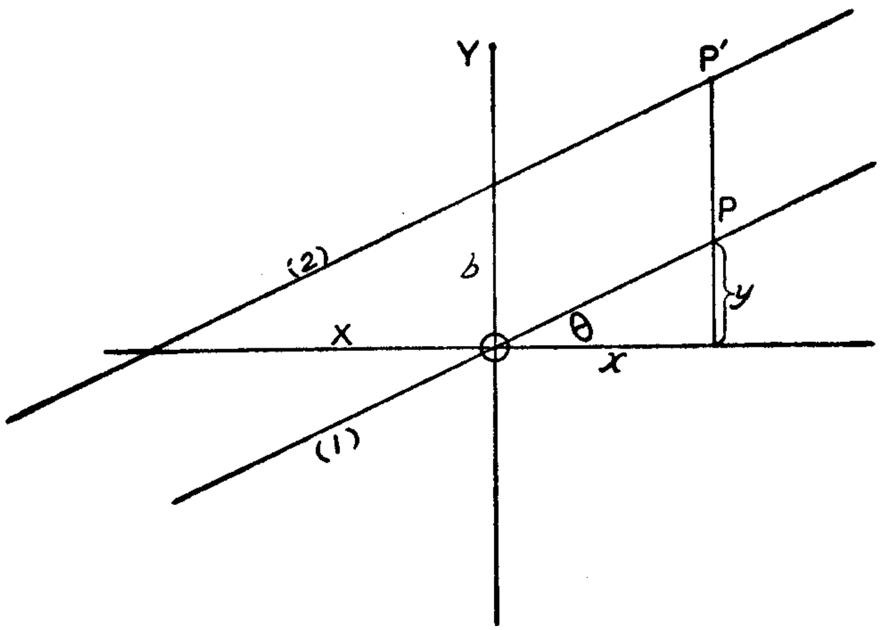

FIG. 10.

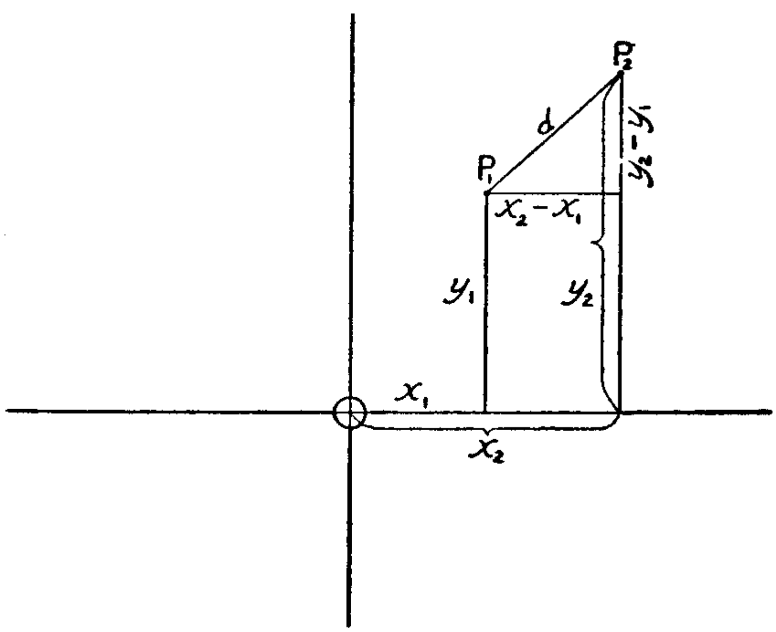

FIG. 11.

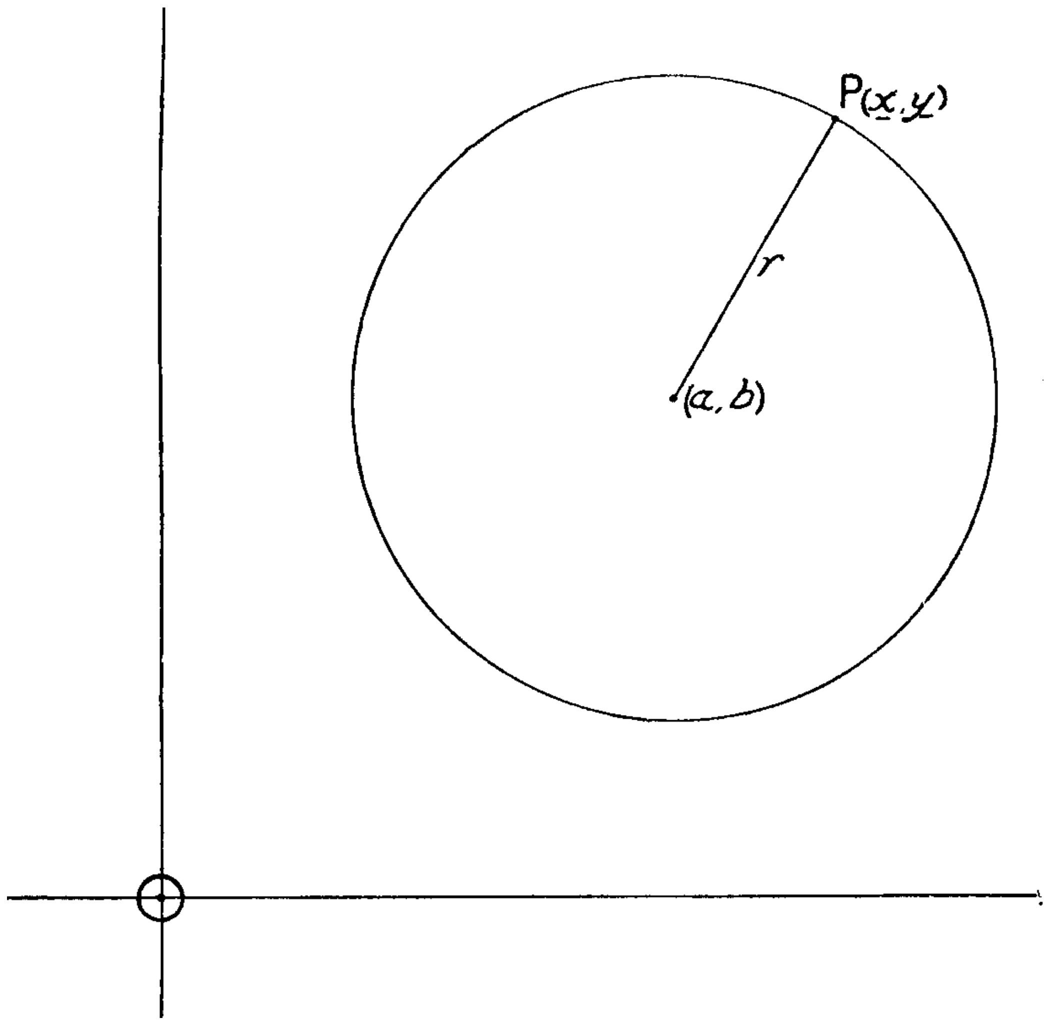

FIG. 12.

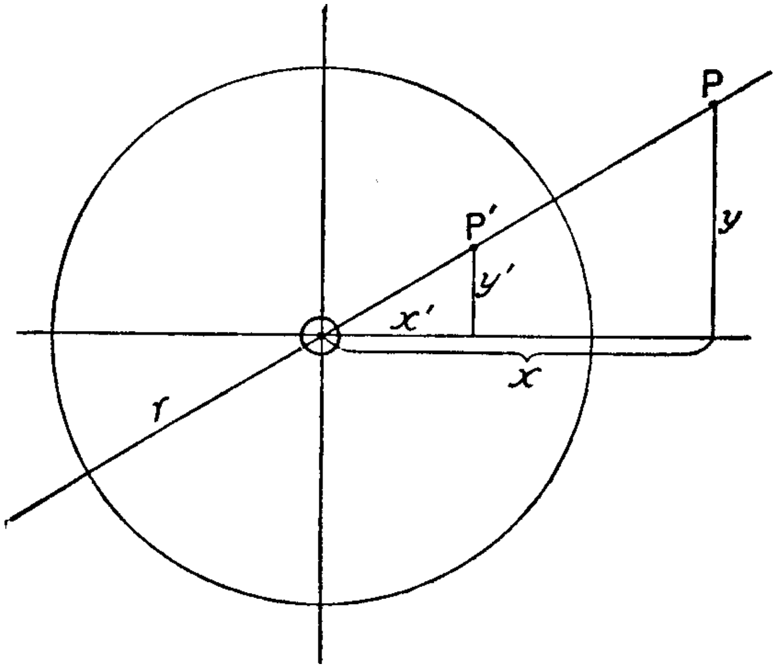

FIG. 13.

From the foregoing distance formula, you see that the *equation* of any circle, Fig. 12, of radius r and center (a, b) is $(x-a)^2+(y-b)^2=r^2$; that is, $x^2+y^2-2ax-2by+a^2+b^2-r^2=O$. Conversely, any equation of the form $x^2+y^2+2Ax+2By+C=O$ represents a circle of center $(-A,-B)$ and squared radius, A^2+B^2-C.

On any line through the center of a circle of radius r let P, P' be such that *distance* OP *times distance* $OP'=r^2$; the point P (or P') is called the *inverse* of P' (or P); the circle and its center are called the inversion circle and center. Taking the circle's center for *origin,* Fig. 13, you will easily find that:

$$(1)\quad \begin{cases} x = \dfrac{x'r^2}{x'^2+y'^2} \\[2ex] y = \dfrac{y'r^2}{x'^2+y'^2} \end{cases} \qquad\qquad (2)\quad \begin{cases} x' = \dfrac{xr^2}{x^2+y^2} \\[2ex] y' = \dfrac{yr^2}{x^2+y^2} \end{cases}$$

Notice that to each point there corresponds one and but one point—except that the inversion center corresponds to no point (in the Euclidean plane). To remove this exception it is common to assume the existence of one and but one "ideal" point, or point at infinity, to serve as the inverse of the center. The new sort of plane thus got is called the *Inversion Plane.* The foregoing point-to-point correspondence is called the *Inversion Transformation* of the plane with respect to the given circle. Clearly, any line through the center is converted into itself. What is the transform, or inverse, of a line not through the center? Let $Ax + By + C = O$ be such a line; replace the coordinates (x, y) of any point in it by their values taken from (1), simplify the result and (if you like) drop the primes; we thus get

$$(3) \quad x^2 + y^2 + r^2\frac{A}{C}x + r^2\frac{B}{C}y = 0$$

This, you note, is a circle through the inversion center, which is here the origin, for the coordinates $(0, 0)$ of the origin satisfy the equation. Hence every line not through the center has for its *transform,* or inverse, a circle through the inversion center.

With these simple ideas held in reserve for use as we need them, let us proceed to our third geometric interpretation. It will be advantageous to deal first with $H\Delta F$. Denote by π an *inversion* plane. Let O be a chosen point of π. The ensemble of all circles through O is called a bundle of circles. The bundle includes, as infinite circles (*i.e.,* circles of infinite radius), the straight lines through O. Now, in thought, let us, once for all, remove the point O from π. Each circle of the bundle now lacks a point; we may call them *pathocircles,* and speak of the O-bundle of *pathocircles.* Our field of operation—which may be denoted by K—is composed of the pathocircles of the O-bundle and the points (except O, of course) of π. We are going to give the doctrinal function $H\Delta F$ an interpretation in the field K; it will be a geometry of certain elements of K. The interpretation arises from assigning to the variable-symbols in the postulates of $H\Delta F$ definite meanings as follows: v_1 will mean a point of K; v_2, a pathocircle; R_1 will mean *between* in the sense that, if A, B, C be three points of a pathocircle, B will be said to be between A and C, if A (or C) must go through B in moving on the pathocircle to C (or A); and R_2 will mean *congruent* in the sense that, if two segments or angles be congruent in the ordinary sense (interpretation D_1), their transforms, or inverses, with respect to a given circle with O as center, will be called congruent in the field of K.

We have now to show that the postulates are verified by the meanings

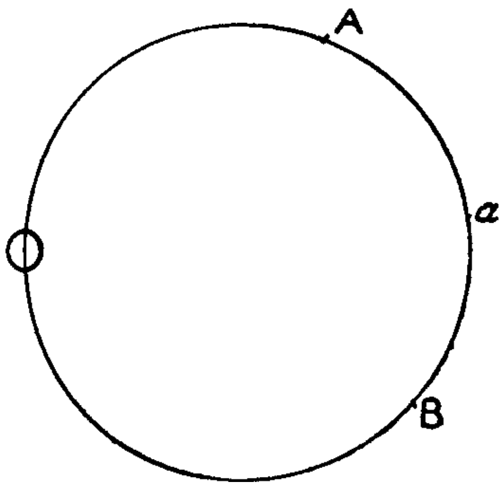

FIG. 14.

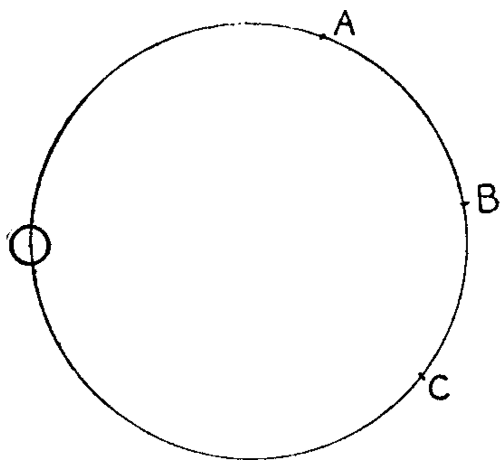

FIG. 15.

assigned. Before giving a proof valid for all of the postulates, it will be instructive to deal with a selected few of them singly by means of simple figures, as in the preceding lecture. Postulate (1) is evidently satisfied. In Fig. 14 the two points *A* and *B* determine the pathocircle *a* of the *O*-bundle.

Fig. 15 exhibits the fact that postulate (8) is verified. Point *B* is between *A* and *C* and between *C* and *A*; neither *A* nor *C* is between the other two of the three points; of course, no point can move through the absent *O*.

Let us next have a look at postulate (12). But we must premise some

DEFINITIONS.—A pair of points, *A* and *B*, of a pathocircle is a *segment* *AB* or *BA*; *A* and *B* are its *ends*; the points between them are the *segment's points.*

It is easy to see, Fig. 16, that the Pasch postulate (12) is verified. *A*, *B*, *C* are three points not on a same pathocircle; they determine three segments, *AB*, *BC*, *CA*; the pathocircle *a* going through *AB*, one of the three, goes through another, *BC*.

Let me suggest that, as an exercise, you make a figure illustrating that the *theorem* (corresponding to Hilbert's theorem 5) dealt with in the preceding lecture, is verified in the present interpretation.

Let us turn to the parallel postulate (13). That it is satisfied is clear in the light of Fig. 17. The given pathocircle is *a*; *A* is a point not on *a*; through *A* there is evidently one and but one pathocircle *b* having no

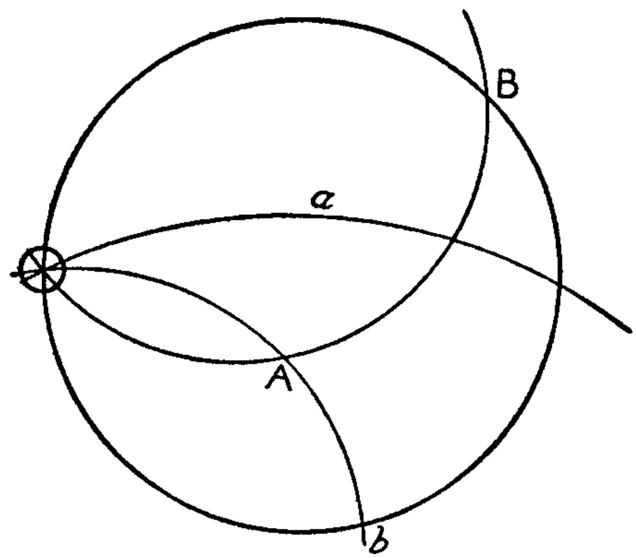

FIG. 16.

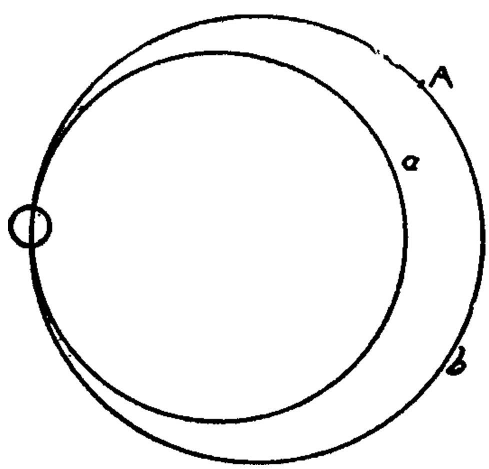

FIG. 17.

point in common with a; a and b are, of course, *parallel* to each other. This postulate, as you know, is *the* Euclidean postulate *par excellence*—the one that mainly distinguishes Euclidean geometry from the famous non-Euclidean geometries of Lobachevski and Riemann. And so you see, in passing, that all interpretations of $H\Delta F$ or $H\Delta F'$ yield doctrines of Euclidean *type*—in the sense that in them the foregoing postulate of parallels is satisfied: they all of them contain some theorems whose proofs depend upon that postulate.

That all of the postulates of $H\Delta F$ are verified by the meanings we have assigned to their variables may be quickly made evident by help of the inversion transformation, explained a little while ago. Let us suppose our field K to be laid down upon a Euclidean plane π. Remember that O is absent from K but that, below the vacant position, π has a point, which we may call O'. In K take a definite circle I for inversion circle having O for center. Regard the transformation as converting the points of K (or π) into the points of π (or K), noting that O' of π and the "ideal" point of K are each the other's transform; that the lines of π are converted into the pathocircles of K, and conversely; and that, if, in π, a point B is between A and C on a line, then in K the transform of B is between the transforms of A and C on a pathocircle, the transform of the line. You see that there is thus established a one-to-one correspondence between the points and lines of π and the points and pathocircles of K, in such a way that all postulated relations among the elements of π hold equally among the corresponding elements of K.

Though logically superfluous, it will be instructive to illustrate the matter a little further by simple figures. In Fig. 18, I is the inversion circle; a is a line in π; pathocircle a' is the transform of a; points A', B', C' are the transforms of A, B, C; segments AB and BC are congruent in the familiar sense—in doctrine D_1; their transforms $A'B'$ and $B'C'$ are congruent in the new sense. You see that the postulate of Archimedes, postulate (20), is verified; for as congruent segments stretch upward in endless succession along a, their congruent transforms proceed on a' in endless succession towards O, never reaching this vacant point-position.

Fig. 19 illustrates congruence of triangles in the new interpretation. Triangles ABC and $A_1B_1C_1$ are congruent in π—in D_1; their transforms,—the new triangles $A'B'C'$ and $A_1'B_1'C_1'$,—are congruent in K—in the new interpretation.

Let us denote the doctrine arising from the new interpretation of $H\Delta F$

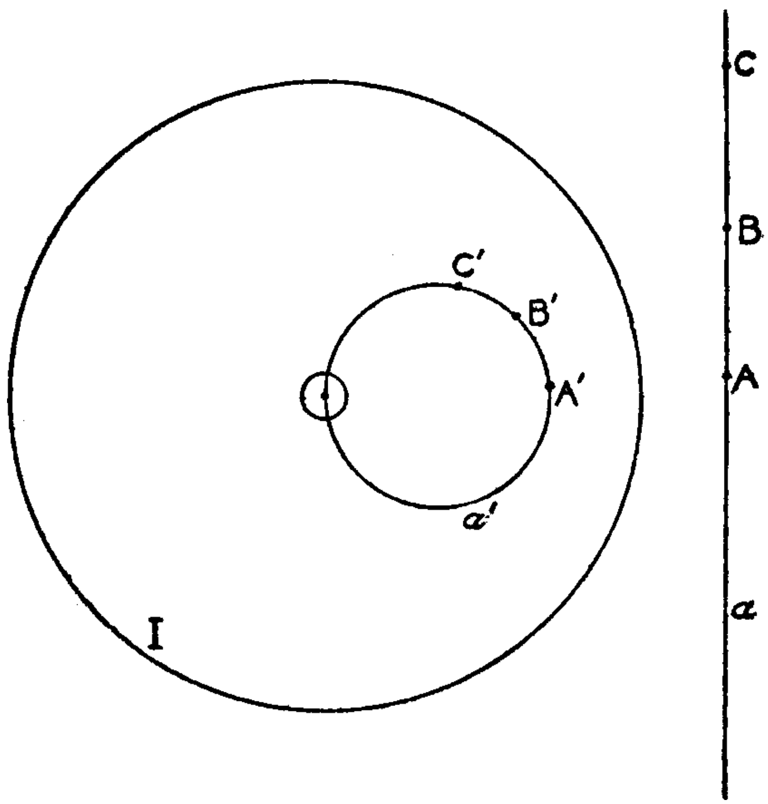

FIG. 18.

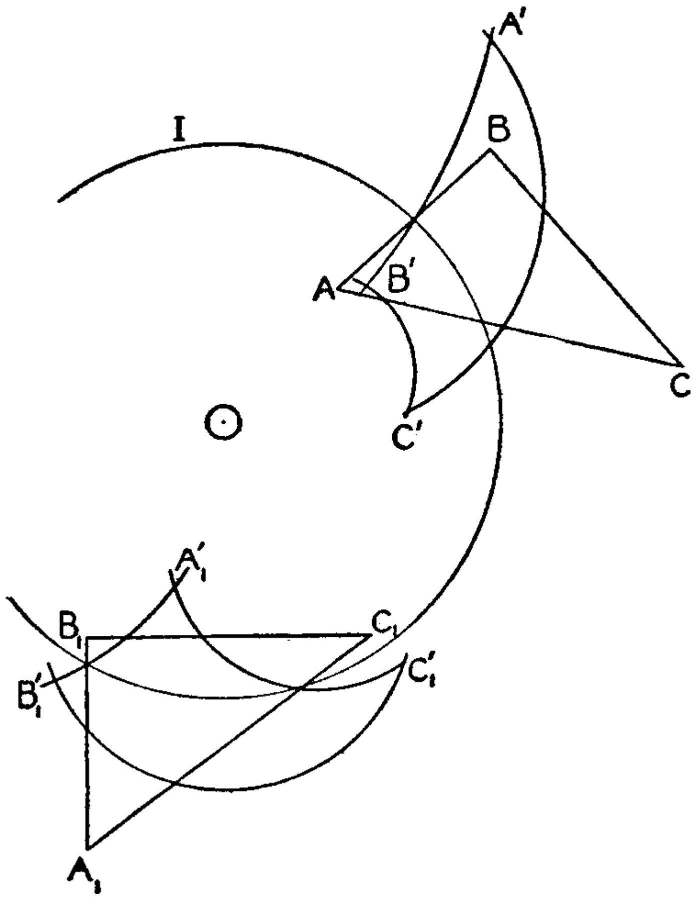

FIG. 19.

by D_3. D_3 is, as you see, a two-dimensional geometry of the points and pathocircles of the field K and is isomorphic with D_1 and D_2. We may say that D_1 is converted, element for element, figure for figure, proposition for proposition, into D_3 by the inversion transformation just as D_1 is completely converted into D_2 by the pole-polar transformation. You thus begin to glimpse the office and power of what mathematicians call transformation, which, at the close of the first lecture, I named, as you will remember, among the pillar-concepts of mathematics.

It remains to give $H\Delta F'$ an interpretation analogous to that we have just given to $H\Delta F$. I will sketch it merely, inasmuch as you will find a fairly full account of it in Weber and Wellstein's *Elementare Geometrie,* which is the second volume of their *Encyklopädie der Elementar-Mathematik*—an excellent work handling in a maturely critical way the various elementary branches of mathematics. You should not be misled by the adjective *Elementare,* for the discussions are designed for advanced students.

By way of a preliminary, I should say a word respecting inversion transformation of ordinary Euclidean space with respect to a given sphere. If the radius be r and the center C, then two points, P and P', on a line through C, are inverses of each other provided the *distance* CP *times the distance* $CP' = r^2$. You easily see that to each point there corresponds one and but one point, except that C has no correspondent (in the Euclidean space). To annul the exception we assume one "ideal" point, or point at infinity, to serve as the transform of the inversion center C. The new space thus obtained is called *inversion space.* The lines and planes through C are transformed into themselves. All lines and planes not through C are converted respectively into circles and spheres through C.

And now for the field of our new interpretation. You probably guess what it is to be. Let S be an inversion space; O a chosen point in it. The ensemble of all the spheres (including planes as spheres of infinite radius) that go through O may be called the O-cluster of spheres. Now remove the point O from S; the cluster is now the O-cluster of pathospheres; and the cluster of circles bereft of O will be called the O-cluster of pathocircles. Our field,—let us denote it by K',—is composed of the points (except O) of S, the pathospheres and pathocircles of the O-clusters.

I need hardly say,—for you doubtless foresee,—that our new interpretation of $H\Delta F'$ springs from agreeing that v_1 shall mean a point of K'; v_2 shall mean a pathocircle; v_3 shall mean a pathosphere; R_1 shall mean between in the sense explained for the field K; and R_2 shall mean congruent in the sense that the transforms of segments or angles congruent in the familiar sense of D_1' shall be congruent in the new sense.

Call the new doctrine thus arising D_3'. It evidently is a three-dimensional geometry of the points, pathocircles and pathospheres of the field K' and matches D_1' or D_2' proposition for proposition. Once more let me emphasize the fact that the differences—the very striking differences—of these three geometries are psychological; logically the three are one.

The next lecture will present a *non*-geometric interpretation of our two doctrinal functions.

Lecture VI

Non-Geometric Interpretation

NOT ALL THAT GLITTERS IS GOLD—A DIAMOND TEST OF HARMONY—TWO-DIMENSIONAL DOCTRINE OF NUMBER DYADS AND SYSTEMS THEREOF—THE THREE-DIMENSIONAL ANALOGUE.

The interpretations, or doctrines, which have hitherto concerned us—D_1, D_2, D_3 of $H\Delta F$ and D_1', D_2', D_3' of $H\Delta F'$—ought to be called, and I have called them, *geometric* doctrines because their content or subject-matter,—that which the doctrines are doctrines of or about,—consists of things, whether sensible or purely conceptual, that are essentially and ultimately *spatial* in kind. The distinction is psychological; mathematicians, not being able to tell precisely what space is, and disdaining or affecting to disdain psychology, may ignore the distinction, if they like—such asininity not being penalized by municipal law in any land. Let us not be so uncandid or so dull as to ignore the essential distinction between spatial and non-spatial doctrines merely because they happen to have the same *form.* Not all that glitters is gold. Let us not so easily lose our common sense—a box of table sugar is not a box of table salt even if the two boxes are identical in size and form.

In the present lecture I invite your attention to a non-geometric interpretation of our doctrinal functions—to an interpretation, or doctrine, to be properly called non-geometric because, though the same in form as the foregoing geometries, it deals with non-spatial things and so has a non-spatial content. Some years ago I asked Mr. Wellington Koo, then a student at Columbia University and a pupil of mine, a brilliant pupil, in analytical geometry, to tell me what the Chinese word for geometry means as a word. He replied: "It means show by a figure." In the interpretation we are about to study we can have no figures, for figures are spatial affairs. This necessity of getting on without figures is, in a sense, fortunate—fortunate as an intellectual discipline—for, in the absence of sensuous representation by figures, we shall be driven to a kind of sheer thinking. And this warning, I hope, will prepare you for the needed effort.

As in the previous lecture, I will deal first with $H\Delta F$. At a later stage of our course, the nature of what is called the *system of real numbers* may be discussed. But for the purposes of the present lecture, I shall assume that you are sufficiently acquainted with the system, merely reminding you that it is composed of the positive and negative integers; the ordinary fractions; the irrationals, such as $\sqrt{2}$, $\sqrt[3]{7}$; and the transcendental numbers, like e and π, for example. By the term number I shall mean a real number. In order to indicate the nature and the field of our new interpretation, it will be convenient to make use of this *definition*: If a, b, c be three numbers, b will be said to be *between* a and c (or c and a) when and only when $a > b > c$ or $a < b < c$, where $>$ means *greater than* and $<$ means *less than.*

The new field of operation—which may be denoted by N—consists of all *dyads* (x, y) of real numbers; that is, of all *ordered pairs* (x, y), where by ordered I mean that (x, y) will not be the same as (y, x) unless $x = y$. It is, of course, understood that the dyads (x_1, y_1) and (x_2, y_2) are distinct unless $x_1 = x_2$ and $y_1 = y_2$. You see that the field is non-spatial, non-geometric, for numbers and number dyads have no essential reference to space and would continue to be perfectly good objects, or subjects, of thought if all spatial sense and all conception of space were to vanish; *symbols* for numbers and for dyads do indeed occupy *room,* but numbers themselves and dyads do not.

And now it is time to say that our non-geometric interpretation of $H\Delta F$ arises from assigning to the postulate variables constant values, or meanings, as follows: v_1 will mean a dyad of N; v_2 will mean a system of dyads, *i.e.,* the dyads satisfying an equation of the form $Ax + By + C = 0$, where either A or B is not zero, *i.e.,* $A \neq 0$ or $B \neq 0$; R_1 will mean between in the sense that, if (x_1, y_1), (x_2, y_2) and (x_3, y_3) are three dyads of a same system, (x_2, y_2) will be said to be between (x_1, y_1) and (x_3, y_3) if and only if x_2 is between x_1 and x_3 or y_2 is between y_1 and y_3, and R_2 will mean congruent in the sense that two dyadic pairs (x_1, y_1), (x_2, y_2) and (x_3, y_3), (x_4, y_4),—that is, two segments $(x_1, y_1)(x_2, y_2)$, $(x_3, y_3)(x_4, y_4)$,—will be said to be congruent when and only when $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_3 - x_4)^2 + (y_3 - y_4)^2}$; with a like meaning for congruence of angles to be give later.

Are the postulates in $H\Delta F$ verified by the meanings thus assigned? It will be very instructive to examine the matter somewhat carefully.

Postulate (1).—Let $Ax + By + C = 0$ be an undetermined system s; d_1, d_2, any two dyads (x_1, y_1), (x_2, y_2) of field N; d_1 and d_2 will belong to s when and only when

$$(1) \quad \begin{cases} Ax_1 + By_1 + C = 0, \\ Ax_2 + By_2 + C = 0; \end{cases}$$

three cases are possible and only three: (α) $x_1 = x_2$, $y_1 \neq y_2$; (β) $x_1 \neq x_2$, $y_1 = y_2$; (γ) $x_1 \neq x_2$, $y_1 \neq y_2$. In (α) $B = 0$ and $\frac{C}{A} = -x_1 = -x_2$; in ($\beta$) $A = 0$, and $\frac{C}{B} = -y_1 = -y_2$; in ($\gamma$) plainly $A \neq 0$, $B \neq 0$, and if $C = 0$, then

$$\frac{A}{B} = -\frac{y_1}{x_1} = -\frac{y_2}{x_2},$$

but if $C \neq 0$, then

$$\frac{A}{C} = -\frac{\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} \quad \text{and} \quad \frac{B}{C} = -\frac{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}};$$

so that in all cases s is determined, and postulate (1) is verified.

Postulate (2).—The question is: do every two dyads of a system determine it? Let $Ax + By + C = 0$ be a system s; (x_1, y_1), (x_2, y_2), any two dyads, d_1, d_2, of s. These dyads, by postulate (1), do determine a system s', say $A'x + B'y + C' = 0$. We are to show that s and s' are the same system.

If $A = 0$, then $A' = 0$, for d_1 is $\left(x_1, -\frac{C}{B}\right)$ and d_2 is $\left(x_2, -\frac{C}{B}\right)$; hence

$$(1) \quad \begin{cases} BA'x_1 - B'C + BC' = 0, \\ BA'x_2 - B'C + BC' = 0, \end{cases}$$

since d_1 and d_2 belong to s'; as $x_1 \neq x_2$, $A' = 0$; so s' is $B'y + C' = 0$; hence $B'y_1 + C' = 0$, and as d_1 belongs to s, $By_1 + C = 0$; so $\frac{C'}{B'} = \frac{C}{B}$, and hence s' is the same system as s. If $B = 0$, the same identity results.

If $A \neq 0$ and $B \neq 0$, then, if $C = 0$, $C' = 0$, for d_1 is $\left(x_1, -\frac{A}{B}x_1\right)$, d_2 is $\left(x_2, -\frac{A}{B}x_2\right)$, and $\frac{A}{B} = -\frac{y_1}{x_1} = -\frac{y_2}{x_2}$; from the last we see that $x_1 \neq x_2$, $y_1 \neq y_2$; from the equation of s' we have

$$(2) \qquad \begin{cases} A'Bx_1 - AB'x_1 + BC' = 0, \\ A'Bx_2 - AB'x_2 + BC' = 0; \end{cases}$$

if $x_1 = 0$ or $x_2 = 0$, then $C' = 0$, as $B \neq 0$; if $x_1 \neq 0$, $x_2 \neq 0$, divide (2) by x_1 and x_2 respectively and then subtract; so it is seen that $C' = 0$. Hence s' is $A'x + B'y = 0$, and, as $A'x_1 + B'y_1 = 0$, $\frac{A'}{B'} = -\frac{y_1}{x_1} = \frac{A}{B}$ and so, again, s' and s are the same.

Finally, if $A \neq 0$, $B \neq 0$ and $C \neq 0$, then, by the foregoing reasoning, $A' \neq 0$, $B' \neq 0$ and $C' \neq 0$. Hence

$$\frac{A'}{B'} = -\frac{\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} = \frac{A}{B} \quad \text{and} \quad \frac{B'}{C'} = -\frac{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} = \frac{B}{C};$$

and so in this, as in all other cases, s and s' are identical. And postulate (2) is verified.

Postulate (7).—This is the next postulate in $H\Delta F$. It is satisfied, for any system $Ax + By + C = 0$ is evidently satisfied by infinitely many dyads, and it is evident that no system contains all the dyads of N.

Postulate (8).—If $d_1(x_1, y_1)$, $d_2(x_2, y_2)$, $d_3(x_3, y_3)$ belong to a same system and if d_2 is between d_1 and d_3, then x_2 is between x_1 and x_3, or y_2 is between y_1 and y_3; then, by definition, x_2 is between x_3 and x_1, or y_2 is between y_3 and y_1, and so d_2 is between d_3 and d_1. Hence the postulate is satisfied.

Postulate (9).—It is evident that in determining a dyad of any given system we can assign the x (or y) at will. Now let $d_1(x_1, y_1)$ and $d_3(x_3, y_3)$ be two given dyads of any given system s; let $d_2(x_2, y_2)$ be a dyad of s

such that $x_1 < x_2 < x_3$; then d_2 is between d_1 and d_3; next let $d_4(x_4, y_4)$ be such that $x_1 < x_3 < x_4$; then d_3 is between d_1 and d_4. Hence the postulate is satisfied.

Postulate (10).—We need consider only four possibilities: (α) $A = 0$, and s is $By + C = 0$; (β) $B = 0$, and s is $Ax + C = 0$; (γ) $A \neq 0$, $B \neq 0$, $C = 0$, and s is $Ax + By = 0$; (δ) $A \neq 0$, $B \neq 0$, $C \neq 0$, and s is $Ax + By + C = 0$.

You know that of three numbers one and only one is between the other two. In (α) any three dyads of s are of the form $\left(x_1, -\frac{C}{B}\right)$, $\left(x_2, -\frac{C}{B}\right)$, $\left(x_3, -\frac{C}{B}\right)$; hence one and only one of the x's is between the other two, and so, too, of the dyads; in (β) like reasoning leads to the same conclusion; in (γ) let $d_1(x_1, y_1)$, $d_2(x_2, y_2)$, $d_3(x_3, y_3)$ be any three dyads of s; then $\frac{A}{B} = -\frac{y_1}{x_1} = -\frac{y_2}{x_2} = -\frac{y_3}{x_3}$; hence no two x's (or y's) are equal for, if they were, the corresponding y's (or x's) would be equal and we should not have three distinct dyads; hence one and only one of the x's (and also one and only one of the y's) is between the other two; hence so, too, the dyads; finally, in (δ) we have

$$\frac{A}{C} = -\frac{\begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix}}{\begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & y_3 \\ 1 & y_1 \end{vmatrix}}{\begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}},$$

since any two of the dyads determine s; if two of the y's were equal, then $A = 0$, contrary to hypothesis, unless the corresponding x's were also equal, but then we should not have three distinct dyads. Hence one and only one of the y's (or x's) is between the other two, and, the same being consequently true of the dyads, the postulate is verified.

The next postulate in $H\Delta F$ is the beautiful postulate (12). First, however, we must have a

Definition.—A pair of dyads, $d_1(x_1, y_1)$ and $d_2(x_2, y_2)$ of an s, is a *segment* d_1d_2 or d_2d_1; d_1 and d_2 are its *ends*; all dyads between the ends are the *segment's dyads.*

Postulate (12).—Let us notice, in the first place, that, taken two at a time, three dyads, $d_1(x_1, y_1)$, $d_2(x_2, y_2)$, $d_3(x_3, y_3)$, not belonging to a same system, determine three systems, s_1, s_2, s_3, as follows:

$$s_1 : \frac{x - x_2}{x - x_3} = \frac{y - y_2}{y - y_3} = \lambda_1,$$

$$s_2 : \frac{x - x_3}{x - x_1} = \frac{y - y_3}{y - y_1} = \lambda_2,$$

$$s_3 : \frac{x - x_1}{x - x_2} = \frac{y - y_1}{y - y_2} = \lambda_3;$$

it is plain that there is but one restriction on the λ's, namely, $\lambda_1 \neq 1$, $\lambda_2 \neq 1$, $\lambda_3 \neq 1$; for, except for the inequalities, the given dyads would not be distinct. Looking at s_1, for example, you see that, when the variable dyad $d(x, y)$ is between d_2 and d_3 (*i.e.,* when it belongs to the segment $d_2 d_3$), λ_1 is *negative*; and that, if λ_1 is *negative* (and neither zero nor ∞), d is in the segment $d_2 d_3$. Clearly the same statement, *mutatis mutandis,* is valid for λ_2 and λ_3.

Solving the foregoing equations for x and y, we get

$$\text{for } s_1 : \begin{cases} x = \dfrac{x_2 - \lambda_1 x_3}{1 - \lambda_1}, \\ y = \dfrac{y_2 - \lambda_1 y_3}{1 - \lambda_1}; \end{cases}$$

$$\text{for } s_2 : \begin{cases} x = \dfrac{x_3 - \lambda_2 x_1}{1 - \lambda_2}, \\ y = \dfrac{y_3 - \lambda_2 y_1}{1 - \lambda_2}; \end{cases}$$

$$\text{for } s_3 : \begin{cases} x = \dfrac{x_1 - \lambda_3 x_2}{1 - \lambda_3}, \\ y = \dfrac{y_1 - \lambda_3 y_2}{1 - \lambda_3}. \end{cases}$$

Now let us suppose that $Ax + By + C = 0$ is a system s not containing any of the dyads d_1, d_2, d_3. The conditions that s shall contain a dyad of each of the systems s_1, s_2, s_3, are respectively

$$\lambda_1 = \frac{Ax_2 + By_2 + C}{Ax_3 + By_3 + C},$$

$$\lambda_2 = \frac{Ax_3 + By_3 + C}{Ax_1 + By_1 + C},$$

$$\lambda_3 = \frac{Ax_1 + By_1 + C}{Ax_2 + By_2 + C};$$

We have, as you see, $\lambda_1\lambda_2\lambda_3 = 1$; hence none of the λ's is negative or else two (and only two) of them are negative. Now suppose that s contains a d in the segment d_1d_2; then λ_3 is negative; hence λ_1 or λ_2 is negative; and so s contains a d of segment d_2d_3 or segment d_1d_3. Hence, you see, our postulate is verified.

Postulate (13).—That this postulate of parallels is verified in our new interpretation may be quickly seen as follows. Let $Ax+By+C = 0$ be any given system s, and let $d(x', y')$ be any dyad not belonging to s. Then *any* system s' containing d is $A'(x-x')+B'(y-y') = 0$, or $A'x+By'+C' = 0$ where $C' = -(A'x' + B'y')$. Solving s and s' for x and y, we get

$$x = -\frac{\begin{vmatrix} C & B \\ C' & B' \end{vmatrix}}{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}}, \quad y = -\frac{\begin{vmatrix} A & C \\ A' & C' \end{vmatrix}}{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}};$$

the two terms of neither fraction can be *zero,* for, if they were, then $\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}$, and s and s' would coincide, contrary to hypothesis; hence x and y have definite finite values and accordingly s and s' have a common dyad (x, y), except when the denominator is zero, but this can happen when and only when $\frac{A'}{B'} = \frac{A}{B}$, and hence there is one and only one s' having no dyad in common with s, this unique s' being parallel to s. And, as you see, the postulate is satisfied.

Before examining postulate (14) we require a

Definition.—If $d'(x', y')$ be a given dyad of a system s, any dyad $d(x, y)$ will be said to be *on the one side or on the opposite side of* d'

according as $x > x'$ or $< x'$, except when s is of the form $Ax + C = 0$ and then the distinction of sides will depend on whether $y > y'$ or $y < y'$.

Postulate (14).—Let $d_1(x_1, y_1)$ and $d_2(x_2, y_2)$ be any segment d_1d_2; let $d'(x', y')$ be any given dyad of any given system s; it is clear that there is in s at least one dyad $d''(x'', y'')$ such that d_1d_2 is congruent with $d'd''$, *i.e.,* such that

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x' - x'')^2 + (y' - y'')^2},$$

for x'' is at our disposal and y'' is a function of it. But is there in s more than one such d''? We know that s is $y = mx + b$ or else $x = m'y + b'$; let us use the former, for the reasoning will be the same as for the latter. If there be a second d'', denote it by $d'''(x''', y''')$, where $x''' = x'' + \delta$; then since d_1d_2, $d'd''$, $d'd'''$ are congruent, we get

$$\sqrt{(x' - x'')^2 + (y' - y'')^2} = \sqrt{(x' - x''')^2 + (y' - y''')^2};$$

note that $y' = mx' + b$, $y'' = mx'' + b$, $y''' = m(x'' + \delta) + b$; substituting these values in the last radical equation, and simplifying, we get $\delta^2 + 2(x'' - x')\delta = 0$; whence $\delta = 0$ or $\delta = 2(x' - x'')$; the former value of δ gives $x'' = x'''$, and so does not give a second d''; the latter value of δ gives $x''' = x'' - 2(x'' - x')$, and so there is one and but one other d''; now note that $x''' - x' = -(x'' - x')$; hence if one d'' is on one side of d', the other d'' is on the other side. And so the postulate is verified.

Postulate (15).—This postulate is so manifestly satisfied that we need not tarry to prove the fact.

Postulate (16).—That this postulate is verified may be readily proved as follows: Let $d_1(x_1, y_1)$, $d_2(x_2, y_2)$ and $d_3(x_3, y_3)$, three dyads of any given system s, be such that the segments d_1d_2 and d_2d_3 have in common no dyad save d_2; let d_1', d_2', d_3', three dyads of any given system s', be such that d_2' is the only dyad common to the segments $d_1'd_2'$ and $d_2'd_3'$. Let d_1d_2 be congruent with $d_1'd_2'$, and d_2d_3 with $d_2'd_3'$; we are to prove that d_1d_3 and $d_1'd_3'$ are congruent. We may take s to be $y = mx + b$, and s'

Lecture XX

Korzybski's Concept of Man

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